A FIRST ANALYTICAL GEOMETRY

JOHN MILNE

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FIRST ANALYTICAL GEOMETRY

(*STRAIGHT LINE AND CIRCLE*)

BY

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PREFACE

THIS little book is intended to provide an introduction to the co-ordinate geometry of "The Straight Line and Circle." In particular the requirements of the Scottish Leaving Certificate are kept in view.

I wish to acknowledge gratefully my indebtedness to Mr. John W. Robertson, M.A., B.Sc., Headmaster, Central Secondary School, Aberdeen, and to Miss Janet Cattanach, M.A., Mathematical Department, Elgin Academy, for much valuable advice and help given during the preparation of the work.

J. M.

April, 1938

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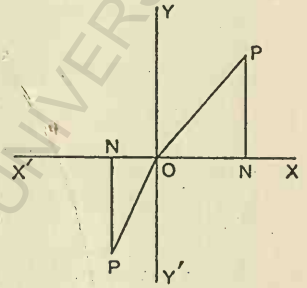
CHAPTER I

CO-ORDINATES: GRAPHS: DIVISION IN A GIVEN RATIO: AREA OF A TRIANGLE

I. *Co-ordinates.* As it is taken for granted that the reader has had a course of graph-drawing, and is therefore acquainted with the groundwork of co-ordinate geometry, no more than a résumé will be given of the elementary conventions of the subject.

Let there be two perpendicular lines $X'X$ (the axis of x) and $Y'Y$ (the axis of y) intersecting at O (the origin).



Let P be any point in the plane of the axes. Draw PN at right angles to $X'X$. Then the position of P will be determined, given the following information,

(i) the lengths of the sides ON and NP of the right-angled triangle ONP,

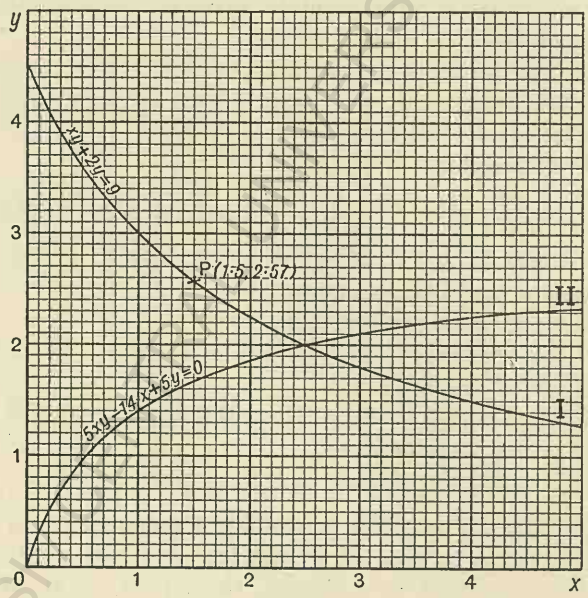
(ii) whether ON be drawn along OX or OX' and NP be drawn up or down.

ON is called the abscissa, NP the ordinate, of P.

The data for marking or plotting a point are given by placing inside a pair of simple brackets two algebraic numbers separated by a comma, thus, $(-1, +3)$. The first number, called the x-co-ordinate of the point, tells the length of ON and, by its sign, whether to draw the segment along OX $(+)$ or along OX' $(-)$. The second number tells the length of NP and by its sign directs whether the perpendicular be drawn up $(+)$ or down $(-)$. To plot the

point $(-1, +3)$, count 1 inch along OX' and 3 inches upwards. To plot the point $(+3, -2)$ count 3 inches along OX and 2 inches downwards, and so on. In general, if P be a point having x as its x-co-ordinate and y as its y-co-ordinate, then we write $P \equiv (x, y)$ or, more simply, $P(x, y)$.

II. *Graphs.* Though it is assumed that the reader is already acquainted with the procedure to be followed in drawing a graph, an illustration will now be discussed.



EXAMPLE 1. *Draw the graph of* $xy + 2y = 9$ *from* $x = 0$ *to* $x = 5$.

The equation gives $y(x + 2) = 9$,

$$\therefore y = \frac{9}{x + 2}.$$

On putting $x = 0, 1, 2 \ldots 5$ in succession we obtain as corresponding values of y the results recorded in the chart.

x	0	1	2	3	4	5
y	4·5	3	2·25	1·8	1·5	1·29

Graph I shows the smooth curve through the plotted points, called the graph of $xy + 2y = 9$.

The following vital features should be apparent :—

(*a*) The corresponding values of x and y in the chart satisfy the given equation simultaneously. The number of these solutions is infinite, there being one for each of the innumerable values we like to name for x (say).

(*b*) To each pair of associated numbers satisfying the equation corresponds a point on the graph.

(*c*) To every point on the graph corresponds a pair of numbers which satisfy the equation.

For example, the diagram shows $P \equiv (1{\cdot}5, 2{\cdot}57)$ and from the equation we derive $y = 2{\cdot}57$ when $x = 1{\cdot}5$.

(*d*) The graph is the locus of points whose co-ordinates satisfy the given equation.

(*e*) A corollary to (*c*) is that in the case of two intersecting graphs the co-ordinates of a common point must satisfy the equation of each graph. In other words, they are a solution of the two equations. Hence, to find the points of intersection of two graphs solve their equations.

EXAMPLE 2. *On the diagram of example* 1 *draw the graph of* $5xy - 14x + 5y = 0$ *from* $x = 0$ *to* $x = 5$, *and name the point of intersection of the two graphs. Compare the co-ordinates with the results obtained by solving the two equations.*

(i) The equation gives $5y(x + 1) = 14x$,

$$\therefore\ y = \frac{14x}{5(x + 1)}.$$

The following chart is now readily compiled :—

x	0	1	2	3	4	5
y	0	1·4	1·87	2·1	2·24	2·33

From the data Graph II is drawn.

The curves are seen to cross at the point (2·5, 2).

(ii) To solve the two equations we put

$$\frac{14x}{5(x+1)} = \frac{9}{x+2} \quad (\text{each side} = y).$$

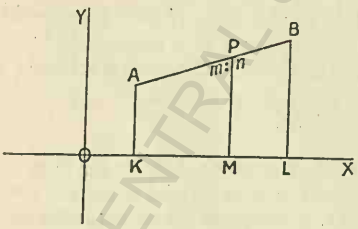
Whence $14x^2 - 17x - 45 = 0$, giving $x = 2\frac{1}{2}$ or $-1\frac{2}{7}$.

From $y = \dfrac{9}{x+2}$ we find $y = 2$ when $x = 2\frac{1}{2}$ and $y = 12\frac{3}{5}$ when $x = -1\frac{2}{7}$. Thus we see that the solutions $x = 2{\cdot}5$, $y = 2$ are the same as the co-ordinates of one of the crossing points.

To verify that $(-1\frac{2}{7}, 12\frac{3}{5})$ is a point of intersection, the graphs would need to be extended, by making a chart for $x = -1, -2, -3$, etc.

III. *The co-ordinates of a point $P \equiv (x', y')$ which divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$.*

Case I. Let P divide AB internally; in other words, let P lie between A and B.



Draw AK, PM and BL perpendicular to $X'X$.

Then by the elementary geometry of the proportional division of lines

$$\frac{KM}{ML} = \frac{AP}{PB} = \frac{m}{n}.$$

$$\therefore \frac{x' - x_1}{x_2 - x'} = \frac{m}{n}, \quad \therefore nx' - nx_1 = mx_2 - mx',$$

$$\therefore (m+n)x' = mx_2 + nx_1,$$

$$\therefore x' = \frac{mx_2 + nx_1}{m+n}.$$

Similarly, by drawing perpendiculars from A, P and B to the y-axis it can be shown that

$$y' = \frac{my_2 + ny_1}{m+n}.$$

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Case II. Let P divide AB externally; in other words, let P lie somewhere on the line not between A and B.

To save drawing, let us imagine that in the figure of Case I, P is interchanged with B and M with L. The reader is now left to prove that $x' = \dfrac{mx_2 - nx_1}{m - n}$ and $y' = \dfrac{my_2 - ny_1}{m - n}$, as an easy exercise.

Instead, however, of treating Case II apart from Case I, we may note that, when the division is external, the segments AP and PB being drawn "with and against the current" so to speak, we may assign to one segment the sign $+$ and to the other the sign $-$. The ratio will now be $m : -n$ (or $-m : n$ if we like), and on changing n to $-n$ in the formulæ of Case I we obtain the formulæ of Case II.

Corollary (important). If P is the mid-point of AB then

$$m = n, \text{ giving } x' = \frac{x_1 + x_2}{2} \text{ and } y' = \frac{y_1 + y_2}{2}.$$

EXAMPLE 1. *Find the co-ordinates of the point which divides the join of* (1, 2) *and* (5, 6) *in the ratio* 4 : 3.

Here $x_1 = 1$, $x_2 = 5$ and $m : n = 4 : 3$,

$$\therefore x' = \frac{(4 \times 5) + (3 \times 1)}{4 + 3} = \frac{23}{7}.$$

Similarly, $$y' = \frac{(4 \times 6) + (3 \times 2)}{7} = \frac{30}{7}.$$

EXAMPLE 2. *Find the co-ordinates of the point which divides the join of* (1, 2) *and* (5, 6) *externally in the ratio* 4 : 3.

As the segments AP and PB are now oppositely directed, we must write $m : n = 4 : -3$,

$$\therefore x' = \frac{(4 \times 5) + (-3 \times 1)}{4 - 3} = 17$$

and $$y' = \frac{(4 \times 6) + (-3 \times 2)}{4 - 3} = 18.$$

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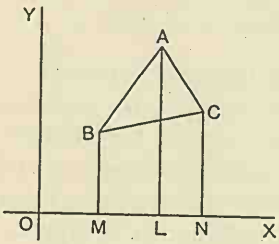
EXAMPLE 3. *Find the co-ordinates of the mid-point of the join of* $(-2, 4)$ *and* $(6, -7)$.

We have $$x' = \frac{x_1 + x_2}{2} = \frac{-2+6}{2} = 2,$$

and $$y' = \frac{y_1 + y_2}{2} = \frac{4-7}{2} = -1{\cdot}5.$$

IV. *Area of the triangle* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.



Draw the perpendiculars AL, BM and CN to the x-axis, and let the area of $\triangle ABC$ be $\triangle$ sq. units.

Then $\triangle ABC$ = figure $ABMNC$ − trapezium $BMNC$.

$\therefore 2\triangle ABC = 2$ (trap. $ABML$ + trap. $ALNC$ − trap. $BMNC$)

$= (LA + MB)ML + (NC + LA)NL - (MB + NC)MN,$

$\therefore 2\triangle = (y_1 + y_2)(x_1 - x_2) + (y_3 + y_1)(x_3 - x_1) - (y_2 + y_3)(x_3 - x_2),$

$$\therefore \triangle = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) \quad (a)$$

$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \quad (b)$$

Notes. (1) For three points in a straight line the value of the formula will be zero, since $\triangle ABC$ then has no area.

(2) The formula is readily quoted and used in the form (a) by writing down the following scheme :—

The terms of (a) within the brackets are the cross products of the pairs of letters in the scheme which are connected by the lines, the pairs linked thus $\diagdown$ having the sign + while the pairs joined thus $\diagup$ have the sign −.

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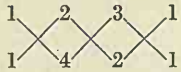
(3) Notice that in the above arrangement the vertices of the triangle corresponding to the co-ordinates come in the order A, B, C, A, that is, in a counter-clockwise circulation round the circum-circle of $\triangle ABC$. If we change the order of the letters and have a clockwise circulation, the formula will give a negative result.

EXAMPLE 1. *Find the area of the triangle* (1, 1), (2, 4), (3, 2).



$$\triangle = \tfrac{1}{2}(4 - 2 + 4 - 12 + 3 - 2) = -2\tfrac{1}{2}.$$

Area of triangle is 2·5 sq. units.

That the vertices have been taken in a clockwise circulation may be confirmed by plotting them.

EXAMPLE 2. *Are* (− 4, − 1), (2, 2) *and* (6, 4) *collinear ?*

$$\triangle = \tfrac{1}{2}(-8 + 2 + 8 - 12 - 6 + 16) = \tfrac{1}{2}(0) = 0.$$

The triangle having no area the points are collinear.

EXAMPLES I

§§ (I-III). 1. Plot the points (2, 3), (− 4, 1), (− 3, − 2), (5, − 1·8), (0, − 2·4), (3·2, 0), (− 2·9, 0).

2. Draw the graphs of

(i) $y = \frac{1}{2}x + 1$ from $x = -2$ to $x = 3$.

(ii) $y = \dfrac{x^2}{2}$ from $x = -2$ to $x = 2$.

(iii) $y = \dfrac{x - 1}{x + 2}$ from $x = -1$ to $x = 4$.

(iv) $xy = 5$ from $x = 1$ to $x = 5$.

(v) $xy - x + 3y + 2 = 0$ from $x = -2$ to $x = 4$.

[See § (II), Example 1.]

Say which of the graphs pass through the origin, and, if possible, where each crosses the axes.

3. By drawing the graphs of $xy = 9$ and $2x - y = 7$ from $x = 1$ to $x = 5$, find a point where the graphs intersect. Verify the result by solving the equations.

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§§ (IV-VIII). 4. Draw on the same diagram the graphs of $y = x^2$ and $y = x + 2$ from $x = -3$ to $x = 3$. Read off the intersections. Verify results by solving the equations.

5. Draw on the same diagram the graphs of $y = 5 - \frac{x^2}{2}$ and $x + 2y = 4$ from $x = -3$ to $x = 4$. Tell the points of intersection and verify the results by solving the equations.

6. Find the mid-points of the joins of the pairs of points :—

(i) (1, 2) and (3, 4).
(ii) (2·2, 3·4) and (5, 4).
(iii) (0, 0) and (− 2, 3).
(iv) (2·6, 0) and (1, 3·8).
(v) (− 3, 2) and (− 1, 4).
(vi) (− 2, − 3·8) and (− 4·6, 0·8).
(vii) (0 − 2·7) and (1·7, 4·9).
(viii) (− 3·6), − 2·4) and (3·6, 2·4).

7. In each case below find the point which divides the join of the given pair of points in the given ratio :—

(i) (2, 2) and (4, 6). Ratio 2 : 3 (internally).
(ii) (− 3, 1) and (2, 5). Ratio 4 : 1 (,,).
(iii) (0, 6) and (4, 0). Ratio 5 : 3 (,,).
(iv) (− 3, − 2) and (1, − 4). Ratio 3 : 5 (,,).
(v) (2·5, 3) and (7, 5). Ratio 2 : 1 (externally).
(vi) (2, 3) and (4, 7). Ratio 7 : 3 (,,).
(vii) (− 4, 2) and (− 5, 2). Ratio 2 : 3 (,,).
(viii) (− 5, 0) and (0, − 7·5). Ratio 1 : 6 (,,).

8. Find the areas of the triangles whose vertices are

(i) (1, 2), (2, 3), (5, 7).
(ii) (− 2, 1), (3, − 2), (5, 7).
(iii) (− 4, 0), (0, 5), (3, − 6).
(iv) (− 4, − 3), (− 1, 6), (5, − 2).
(v) (0, 0), (x_1, y_1), (x_2, y_2).

Harder Examples

9. In the equation $2x + 5y = 10$ what is the value of x when $y = 0$ and the value of y when $x = 0$? Name the points at which the graph crosses the axes.

10. Where does the graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ cross the axes ?

11. Find the area of the triangle having as vertices the points where the graph of $y = x^2 - 4x + 3$ crosses the axes.

12. Find the intersections of $y = 3 - x$ and $y = x^2 - x - 6$.

13. Show that (2, 1) is a point on the graph of $x^2 + y^2 = 5$.

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14. If (8, 2) lies on $xy = c^2$, what is the value of c ?

15. If (2, − 6) lies on $y = mx + 2$, find the value of m.

16. State the condition that (x_1, y_1) lie on $2x + 5y = 7$.

17. If $4x_1 - y_1 = 7$, write down the equation of a graph on which (x_1, y_1) is a point.

18. If $P(x_1, y_1)$ is such that $2x_1^2 - 3x_1y_1 + y_1^2 = 10$ always, what is the equation of the locus of P ?

19. If (2, 5) is the mid-point of a line one of whose extremities is at the point (− 5, 6), find the other extremity.

20. By calculating the area of the triangle whose vertices are (1, 5), (4, 8) and (6, 10) show that these points are collinear. Then find the ratio $m : n$ in which (4, 8) divides the join of the other two points.

21. In the triangle whose vertices are $A(-1, 5)$, $B(2, 3)$ and $C(5, 7)$, find D, E and F the mid-points of BC, CA and AB respectively. Then find the points which divide AD, BE and CF in the ratio 2 : 1. What conclusion do you draw ?

22. Find the centroid (median centre) of the triangle whose vertices are (− 5, − 2), (3, 4) and (− 1, 7).

23. Use the fact that the centroid is a point of trisection of a median, to show that the median centre of the triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is at the point

$$[\tfrac{1}{3}(x_1 + x_2 + x_3), \tfrac{1}{3}(y_1 + y_2 + y_3)].$$

24. Find the median centre of the triangle formed by the points where the graph of $y = 18 + 3x - x^2$ crosses the axes.

25. Four successive vertices of two quadrilaterals are

(i) (− 2, 1), (1, 2), (4, − 1), (1, − 2).
(ii) (− 5, 12), (− 2, − 3), (9, − 10), (6, 5).

Show that each figure is a parallelogram by proving that the joins of opposite vertices have the same mid-point.

26. Three vertices of the parallelogram $ABCD$ in their order are $A(2, 1)$, $B(3\ 5)$ and $C(6, 3)$. Find the co-ordinates of D.

27. (1, 1), (6, 3), (4, 7) and (2, 5) are successive corners of a quadrilateral. Calculate the area by dividing the figure into two triangles.

28. Find the mid-points of the sides of the quadrilateral of example 27. Apply the method of example 25 to prove that these mid-points are the corners of a parallelogram. Show that the area of the quadrilateral is twice that of the parallelogram.

29. From a variable point $P(x, y)$ perpendiculars PN and PM are drawn to the axes. If the area of the rectangle $ONPM$ is always 5 sq. units, find the equation which connects x and y.

30. A is the point (10, 0). $P(x, y)$ is a variable point, and PM is perpendicular to the y-axis. If the square on PM is equal in area to $\triangle OAP$, draw the graph of the equation which gives the locus of P.

31. Use the formula for the area of a triangle (§ IV) to show that the area of the quadrilateral whose vertices in succession are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) is

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4).$$

32. Prove, by applying the formula for the area of a triangle to any two sets of three from among the following points $(-4, 2)$, (2, 5), (4, 6) and (8, 8), that the four points are collinear. Then show that (2, 5) and (8, 8) divide the join of the remaining pair internally and externally in the same ratio.

33. $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ are four vertices of a quadrilateral in order. The mid-points of AB, BC, CD, DA, AC and DB are P, Q, R, S, T and U respectively. By calculating co-ordinates, show that PR, QS and TU have a common mid-point.

34. (i) Find the points where the graph of

$$x^2 + y^2 - 8x - 7y + 12 = 0 \text{ crosses the axes.}$$

(ii) Show that the four points so obtained are concyclic.

(iii) Find the circum-centre of the cyclic quadrilateral.

35. If P, Q, R and S are the mid-points of the sides of the cyclic quadrilateral in example 34, prove that the intersection of the diagonals of the parallelogram which the points define lies mid-way between the origin and the centre of the circle.

36. Show that $X'X$ touches the circum-circle of the triangle whose vertices are the points where the graph of

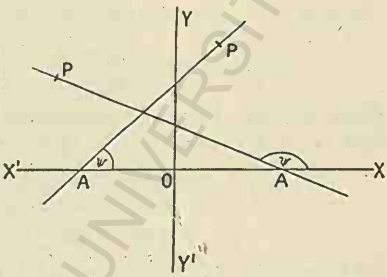
$$y^2 - 6x - 13y + 36 = 0 \text{ crosses the axes.}$$

CHAPTER II

INCLINATION AND GRADIENT: GRAPH OF $lx + my + n = 0$: VARIOUS FORMS OF THE STRAIGHT-LINE EQUATION: PARALLEL LINES, ETC.

I. *The inclination and the gradient of a straight line.*

Inclination. If a point P lying in the first or second quadrants be taken on a straight line which cuts the x-axis at A, then $X\hat{A}P$ is called the inclination of the line.



The angle XAP is commonly named ψ (Greek letter psi).

It may be acute or obtuse.

$X'\hat{A}P$ must not be taken as the inclination. It is the supplement of the inclination.

Gradient. $\tan\psi$ is called the gradient of the line.

II. *Gradient of the line joining the points* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$.

Let the line joining the points P and Q cut the x-axis at A.

Draw PN and QM each perpendicular to the x-axis.

Draw PL parallel to the x-axis (Fig. I, p. 12).

Fig. I. Since $\psi = X\hat{A}P = \hat{LPQ}$,

$$\therefore \tan\psi = \frac{LQ}{PL} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Note. When ψ is obtuse the value of $\dfrac{y_2 - y_1}{x_2 - x_1}$ will

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necessarily be negative. A figure will show that then y_1 is greater than y_2 (Fig. II). (x_1 is assumed $< x_2$.)

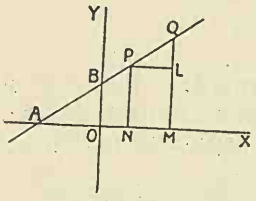

FIG. I.

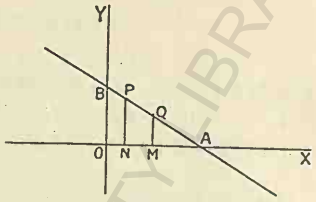

FIG. II.

EXAMPLE 1. *Find the gradient of the join of* (2, 6) *and* (7, 9).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 6}{7 - 2} = 0{\cdot}6.$$

EXAMPLE 2. *Find the inclination of the line joining the points* $(-2, 1{\cdot}7)$ *and* $(-0{\cdot}5, 1)$.

$$\tan \psi = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 1{\cdot}7}{-0{\cdot}5 + 2} = -0{\cdot}4667,$$
$$\therefore \psi = 180° - 25° 1' = 154° 59'.$$

EXAMPLE 3. *If the gradient of the line joining the points* $P(x, y)$ *and* $C(3, 4)$ *be* $\frac{2}{5}$, *find the algebraic relation between* x *and* y.

Since the gradient of $CP = \dfrac{y - 4}{x - 3}$,

$$\therefore \frac{y - 4}{x - 3} = \frac{2}{5}, \text{ whence } 2x - 5y + 14 = 0.$$

Note. The reader should remark that the above is an example of an equation of the first degree in x and y. In such equations there are no products of x and y and no powers of these variables higher than the first. The numbers in the equations, apart from x and y, are given quantities.

Other examples of these first-degree equations are

$$x + 6y - 3 = 0,$$
$$x = 4y, \text{ i.e., } x - 4y = 0,$$
$$x = -7, \text{ i.e., } x + 0y + 7 = 0.$$

The general form is $lx + my + n = 0$ where l, m and n are known numbers.

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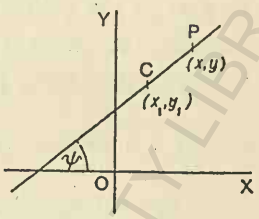
III. *The graph of any equation of the first degree in x and y is a straight line.*

Let $lx + my + n = 0$ be a given equation of the first degree in x and y.



Then if $C \equiv (x_1, y_1)$ be an assigned point on the graph and $P \equiv (x, y)$ be any other point thereon, we have

$$lx + my + n = 0$$

and $$lx_1 + my_1 + n = 0,$$

$$\therefore l(x - x_1) + m(y - y_1) = 0 \text{ (by subtraction)},$$

$$\therefore \frac{y - y_1}{x - x_1} = -\frac{l}{m}. \quad . \quad . \quad . \quad (a)$$

Thus, if ψ be the inclination of CP, then

$$\tan \psi = -\frac{l}{m}, \text{ a constant},$$

$\therefore \psi$ is a definite angle.

Hence $P \equiv (x, y)$, any point on the graph, lies on a straight line of known inclination drawn through the fixed point C. Now one, and only one, line can be drawn under these conditions. Hence the locus of P, or the graph of $lx + my + n = 0$, is a straight line.

For the sake of future exercises it should be noted that the gradient of the straight line is $-\frac{l}{m}$. [See (*a*) above.]

EXAMPLE 1. *Show how to draw the graph of* $2x - 3y + 6 = 0$. It is enough to find two points on the line. Now $y = 4$ when $x = 3$ and $y = 2$ when $x = 0$, hence the graph is the straight line which passes through the points (0, 2) and (3, 4).

Note. The two points chosen are often the intersections of the line with the axes. It is to be noticed that $y = 0$ where the graph cuts $X'X$ and $x = 0$ where it cuts $Y'Y$.

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In the example just worked out we have $x = -3$ when $y = 0$ and $y = 2$ when $x = 0$, showing that the line crosses the axes at the points $(-3, 0)$ and $(0, 2)$.

EXAMPLE 2. *What is the gradient of the line* $2x = 7y + 5$?

Since the gradient of $lx + my + n = 0$ is $-\frac{l}{m}$, the gradient of $2x = 7y + 5$, or $2x - 7y - 5 = 0$, is $-\frac{2}{-7} = \frac{2}{7}$.

IV. *Particular cases of the equation* $lx + my + n = 0$.

(i) Suppose that $n = 0$, but that l and m are not zero. The equation then reduces to $lx + my = 0$. It is plainly satisfied by the values $x = 0$ and $y = 0$. **Hence the graph of $lx + my = 0$ is a straight line passing through the origin.**

(ii) Let $l = 0$, but let m and n not be zero. The equation then becomes $my + n = 0$,

or
$$y = -\frac{n}{m}.$$

The y-co-ordinates of points on this graph are therefore the same in value whatever the x-co-ordinates may be.

The graph is a straight line parallel to $X'X$.

For example, if the equation $y = 3$ be taken, we have

x	-2	-1	0	1	2	3
y	3	3	3	3	3	3

The reader is left to plot these points and so to see for himself the nature of the graph.

(iii) In a similar manner it may be shown that **the graph of $lx + n = 0$ is a straight line parallel to $Y'Y$.**

The equations $y = 0$ and $x = 0$ are evidently those of the x and y-axes respectively.

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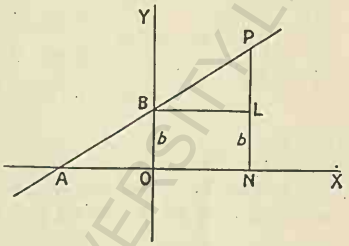
V. *Various forms of the equation of a straight line.*

Sometimes the constants in the equation of a straight line are so designed that they give at once certain geometrical information.

(1) *Equation of a straight line in "Gradient Form."*

In this form the equation reveals the gradient (m) of the line and the intercept (b) on the y-axis.

Let the line cut the y-axis at $B(o, b)$. Let $P(x, y)$ be any other point on the line. Draw PN perpendicular to, and BL parallel to, the x-axis.



Then, since $\tan \hat{LBP} = \tan \hat{XAP}$,

$$\text{the gradient of the line} = \frac{LP}{BL} = m,$$

$$\therefore \frac{NP - NL}{ON} = m,$$

$$\therefore \frac{y - b}{x} = m,$$

whence $$y = mx + b.$$

EXAMPLE 1. *Write down the equation of a straight line of gradient $\frac{2}{3}$ which makes a negative intercept of 4 units on the y-axis [i.e., cuts $Y'Y$ at $(0, -4)$].*

In this case $m = \frac{2}{3}$ and $b = -4$. Hence the equation $y = mx + b$ gives $y = \frac{2}{3}x - 4$ or $2x - 3y - 12 = 0$.

EXAMPLE 2. *Put the equation $5x - 4y + 6 = 0$ into gradient form.*

The equation may be written $4y = 5x + 6$, or $y = \frac{5}{4}x + \frac{3}{2}$, showing that the gradient of the line is $\frac{5}{4}$, and that the intercept made on the y-axis is positive and equal to $\frac{3}{2}$.

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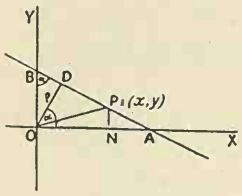
(2) *Equation of a straight line in "Intercept Form."*

In this form the equation shows immediately where the straight line crosses the axes.



Let it cross the axes at

$$A \equiv (a, o) \text{ and } B \equiv (o, b).$$

Let $P \equiv (x, y)$ be any point on the line. Draw PN perpendicular to $X'X$.

Then $\triangle OPB + \triangle OPA = \triangle OAB$,

$\therefore\ OB \,.\, ON + OA \,.\, NP = OA \,.\, OB.$

That is $$bx + ay = ab, \text{ or } \frac{x}{a} + \frac{y}{b} = 1.$$

EXAMPLE 1. *Find the equation of the straight line which crosses the axes at the points* (7, 0) *and* (0, − 2).

Putting $a = 7$ and $b = -2$ in the equation $\frac{x}{a} + \frac{y}{b} = 1$, we have $\frac{x}{7} + \frac{y}{-2} = 1$, whence $2x - 7y = 14$.

EXAMPLE 2. *Arrange the equation* $3x + 4y + 12 = 0$ *in intercept form.*

We have $$-3x - 4y = 12.$$

On dividing both sides by 12 so that the right-hand side may be 1, we obtain $\frac{x}{-4} + \frac{y}{-3} = 1$.

(3) *Equation of a straight line in "Perpendicular Form."*

In this form the equation shows the length of the perpendicular from the origin to the line and the angle (measured counter-clockwise) which the perpendicular makes with OX. (Use the diagram above.)

Draw OD perpendicular to the line and let $OD = p$. Let $X\hat{O}D = \alpha$. Then, as before, $\triangle OPB + \triangle OPA = \triangle OAB$, whence

$$x \,.\, OB + y \,.\, OA = p \,.\, AB,$$

$$\therefore\ x\frac{OB}{AB} + y\frac{OA}{AB} = p,$$

$\therefore\ x \cos\alpha + y \sin\alpha = p$ (since $O\hat{B}A = 90° - D\hat{O}B = \alpha$).

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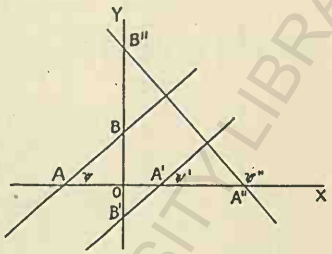
VI. *Parallel and perpendicular straight lines.*

(i) *If the straight lines* $y = mx + b$ *and* $y = m'x + b'$ *are parallel, then* $m = m'$.



If the lines AB and $A'B'$ in the figure are parallel, then must $\psi = \psi'$,

$\therefore \tan\psi = \tan\psi'$,

$\therefore \quad m = m'$.

From this result we can deduce the condition that the lines $lx + my + n = 0$ and $l'x + m'y + n' = 0$ be parallel.

Since the gradients must be equal,

$$\therefore -\frac{l}{m} = -\frac{l'}{m'}, \text{ or } lm' - l'm = 0.$$

EXAMPLE. *Prove that the lines* $2x - y + 5 = 0$ *and* $6x - 3y + 7 = 0$ *are parallel.*

The reader can prove easily that the gradients are equal. Alternatively: $lm' - l'm = (2 \times -3) - (6 \times -1) = 0$.

(ii) *If the straight lines* $y = mx + b$ *and* $y = m''x + b''$ *are perpendicular, then* $mm'' = -1$.

Let $A''B''$ be perpendicular to AB. Then

$$\tan\psi'' = -\tan A\hat{A}''B'' = -\cot\psi = -\frac{1}{\tan\psi},$$

$$\therefore \tan\psi\tan\psi'' = -1 \text{ or } mm'' = -1.$$

In other words, **the product of the gradients must be -1.**

The condition that the lines $lx + my + n = 0$ and $l''x + m''y + n'' = 0$ be perpendicular readily follows.

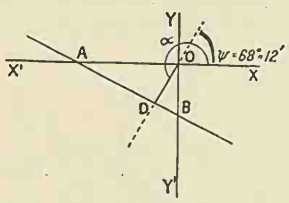
Product of gradients $= -\frac{l}{m} \times -\frac{l''}{m''} = -1$.

Hence we must have $ll'' + mm'' = 0$.

EXAMPLE 1. *Prove that the lines* $3x + 4y - 7 = 0$ *and* $8x - 6y + 5 = 0$ *are perpendicular.*

It is easy to prove that the product of the gradients is -1. Alternatively $ll'' + mm'' = (3 \times 8) + (4 \times -6) = 0$.

EXAMPLE 2. *If OD be perpendicular to $2x + 5y + 3 = 0$, find the angle α which OD makes with OX.*



Since the gradient of AB or $2x + 5y + 3 = 0$ is $-\frac{2}{5}$ that of OD is $\frac{5}{2}$. Hence the inclination of OD is 68° 12′. Now $2x + 5y + 3 = 0$ makes negative intercepts on the axes, and therefore AB lies as drawn in the figure. Hence,

$$\alpha = 180° + 68°\ 12' = 248°\ 12'.$$

VII. *The equation* (i) *of the line which passes through the points $H(x_1, y_1)$ and $K(x_2, y_2)$, and* (ii) *of the line which passes through the point $H(x_1, y_1)$ and has a gradient m.*

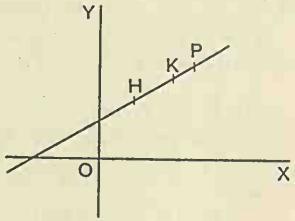
Let $P \equiv (x, y)$ be any point on the line,

Case (i). $\because$ gradient HP = gradient HK (same line),

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which may also be written

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$



Case (ii). Gradient of $HP = m$,

$$\therefore \frac{y - y_1}{x - x_1} = m, \quad \therefore y - y_1 = m(x - x_1).$$

EXAMPLE 1. *Find the equation of the straight line through the points* (1, − 5) *and* (− 2, 4).

Case (i) gives $\dfrac{y + 5}{4 + 5} = \dfrac{x - 1}{-2 - 1}$, or $3x + y + 2 = 0$.

EXAMPLE 2. *Find the equation of the line through the point* (2, − 3) *which is parallel to the line* $2x - 4y = 3$.

The gradient of $2x - 4y = 3$ being $\frac{2}{4}$ or $\frac{1}{2}$, that of a parallel line must also be $\frac{1}{2}$. Hence, in Case (ii) we put $x_1 = 2$, $y_1 = -3$ and $m = \frac{1}{2}$, giving $y + 3 = \frac{1}{2}(x - 2)$, whence $x - 2y = 8$.

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VIII. (i) *Three points* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ *in a straight line.*

(ii) *Three lines passing through the same point.*

(i) If ABC is a straight line, the gradient of AC is that of AB,

$$\therefore \frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Remarks. (α) The above result is tantamount to stating that the area of $\triangle ABC$ is zero, as can be shown by cross multiplying and collecting terms to one side. [Chap. I, § IV, Form (a).]

(β) The result can also be obtained by stating the condition that the point (x_3, y_3) lies on the line AB. [Chap. II, § VII.]

EXAMPLE. *Are* $(-4, -1)$, $(1, 1)$ *and* $(6, 3)$ *collinear points?*

Since

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{3 + 1}{6 + 4} = \frac{2}{5},$$

and

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 + 1}{1 + 4} = \frac{2}{5},$$

$$\therefore \frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence the points are in a straight line.

(ii) *To test whether three lines are concurrent we solve the equations of two of them and then find whether the third equation is satisfied by the results.*

EXAMPLE. *Show that the lines* $x + 3y = 7$, $3x - 2y = 10$ *and* $4x - 5y = 11$ *are concurrent.*

On solving the first two equations, we have $x = 4$ and $y = 1$. These values satisfy the third equation $4x - 5y = 11$. Hence $(4, 1)$ is a point on all three lines.

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IX. *The system of lines*

$$l_1x + m_1y + n_1 + k(l_2x + m_2y + n_2) = 0.$$

When the parts $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$ of the above equation are each zero simultaneously, then the left-hand side of the equation is also zero for $0 + k(0) = 0$. Hence the equation is satisfied by the joint solutions of

$$l_1x + m_1y + n_1 = 0 \quad . \quad . \quad . \quad (1)$$

and

$$l_2x + m_2y + n_2 = 0 \quad . \quad . \quad . \quad (2)$$

that is, by the co-ordinates of the point of intersection of the lines, of which (1) and (2) are the equations. But $l_1x + m_1y + n_1 + k(l_2x + m_2y + n_2) = 0$ is itself the equation of a straight line, since it is of the first degree in x and y. Hence its graph is some straight line through the intersection of (1) and (2). Different straight lines are obtained by changing the value of k, which is called the "parameter" of the system. Thus the equation represents a system or "pencil" of concurrent lines.

EXAMPLE. *Find the equation of a straight line through the intersection of* $x - 2y - 1 = 0$ *and* $3x - 2y - 11 = 0$, (i) *when the line passes through the origin;* (ii) *when it makes a positive intercept of 2 units on the y-axis.*

$x - 2y - 1 + k(3x - 2y - 11) = 0$ is the equation of a line through the intersection of the given lines.

(i) O is on the line if last equation is satisfied by $x = 0$ and $y = 0$, that is, if $-1 + k(-11) = 0$, or $k = -\frac{1}{11}$.
Hence $2x - 5y = 0$ is the required equation.

(ii) To find the intercept on the y-axis put $x = 0$, giving

$$-2y - 1 + k(-2y - 11) = 0 \quad \text{or} \quad -2(1 + k)y = 11k + 1,$$

$$\therefore y = -\frac{11k + 1}{2(1 + k)},$$

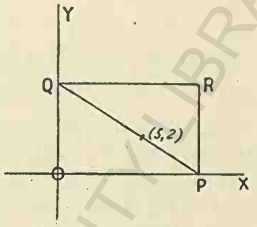
$$\therefore -\frac{11k + 1}{2(1 + k)} = 2, \text{ whence } k = -\tfrac{1}{3},$$

$$\therefore x - 2y - 1 - \tfrac{1}{3}(3x - 2y - 11) = 0,$$

or $y = 2$ is the required equation.

X

EXAMPLE 1. *The figure shows a straight line drawn through the point* (5, 2) *to cut the axes at P and Q. Find the locus of R, the remaining vertex of the rectangle of which OP and OQ are sides.*



Let $R \equiv (x', y')$. Then $OP = x'$ and $OQ = y'$. The equation of PQ is therefore $\frac{x}{x'} + \frac{y}{y'} = 1$.

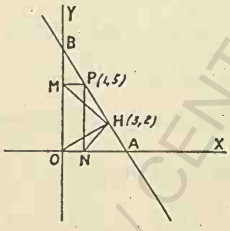
Now (5, 2) is a point on PQ,

$$\therefore \frac{5}{x'} + \frac{2}{y'} = 1,$$

$$\therefore 2x' + 5y' = x'y'.$$

The locus of $R(x', y')$ is therefore the graph of the equation $2x + 5y = xy$.

EXAMPLE 2. *A straight line AB is drawn through the point* $H(3, 2)$ *perpendicular to OH. Find its equation, and show that the point* $P(1, 5)$ *lies on the line.*



(i) Let the equation of AB be

$$y - 2 = m(x - 3).$$

Now the gradient of $OH = \frac{2}{3}$,

$$\therefore m = -\tfrac{3}{2},$$

$$\therefore y - 2 = -\tfrac{3}{2}(x - 3),$$

or $3x + 2y = 13$ is the equation of AB.

(ii) $P(1, 5)$ is plainly a point on $3x + 2y = 13$.

EXAMPLE 3. *In example* 2 *draw PM and PN perpendicular to the axes. Prove HM at right angles to HN.*

We have $M \equiv (0, 5)$, $N \equiv (1, 0)$ and $H \equiv (3, 2)$.

$\therefore$ gradient of $MH = -1$, and gradient of $NH = 1$.

Hence, product of gradients being -1, $\widehat{NHM} = 90°$.

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EXAMPLE 4. *Find the locus of the point* $P(5t - 2,\ 3t - 7)$ *where* t *is a variable.*

Let $P \equiv (x, y)$. Then $x = 5t - 2$ and $y = 3t - 7$.

Whence $3x = 15t - 6$,

and $5y = 15t - 35$.

Subtract $\therefore \quad 3x - 5y = 29$.

Hence the locus of $P(x, y)$ is the straight line $3x - 5y = 29$.

EXAMPLE 5. *Find where the locus of* P (*example* 4) *crosses the straight line* $x - 2y = 11$.

If (x, y), a point on $x - 2y = 11$, is also a point on the locus of P, then must $x = 5t - 2$ and $y = 3t - 7$.

Substitution in the equation of this example gives

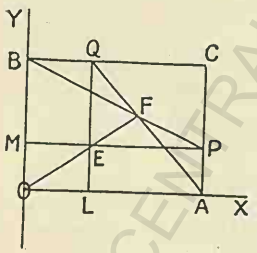
$$(5t - 2) - 2(3t - 7) = 11, \text{ whence } t = 1,$$

$$\therefore\ x = 5t - 2 = 3$$

and $$y = 3t - 7 = -4.$$

The crossing point is, therefore, $(3, -4)$, a result which should be verified by solving the equations $3x - 2y = 29$ and $x - 2y = 11$ in the ordinary way.

EXAMPLE 6. *In the accompanying figure* $C \equiv (6, 3)$ *and* $E \equiv (2, 1)$. *Prove that* OE, BP *and* AQ *are concurrent.*



Plainly $A \equiv (6, 0)$, $B \equiv (0, 3)$, $P \equiv (6, 1)$ and $Q \equiv (2, 3)$,

Hence the equation of OE is

$$\frac{y}{1} = \frac{x}{2} \text{ or } x = 2y, \quad . \quad (1)$$

the equation of AQ is

$$\frac{y}{3} = \frac{x - 6}{2 - 6} \text{ or } 3x + 4y = 18, \quad (2)$$

and the equation of BP is

$$\frac{y - 3}{1 - 3} = \frac{x}{6} \text{ or } x + 3y = 9 \quad . \quad . \quad (3)$$

On solving (1) and (3) we find $x = \frac{18}{5}$ and $y = \frac{9}{5}$. OE and BP, therefore, intersect at the point $F\ (\frac{18}{5}, \frac{9}{5})$. Substitution for x and y now gives $3x + 4y = \frac{54}{5} + \frac{36}{5} = 18$.

Hence the graph of (2) also passes through F. In other words, OE, BP and AQ are concurrent.

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EXAMPLES II

§§ (I-VII). 1. Find the gradients of the lines joining the following pairs of points :—

(i) (1, 2) and (4, 3).
(ii) (3, 11) and (7, 5).
(iii) (− 2, 4) and (5, − 8).
(iv) (− 1, − 6) and (7, 2).
(v) (0, 0) and (5, 4).
(vi) (− 5, − 4) and (0, 0).
(vii) (0, 0) and (x_1, y_1).
(viii) $(a, 0)$ and $(0, b)$.

2. Find the inclination of the lines joining the pairs of points in (i), (ii), (iii) and (iv) of example (1).

3. Draw the straight lines which are the graphs of the following equations :—

(i) $y = 2x - 3$.
(ii) $y = -\frac{1}{3}x + 1$.
(iii) $2x - 3y = 6$.
(iv) $5x - 2y - 4 = 0$.

4. What are the gradients of the following lines :—

(i) $4x + 2y = 5$.
(ii) $2x - 3y + 12 = 0$.
(iii) $5x = 6y$.
(iv) $x \cos \alpha + y \sin \alpha = p$.

5. Prove that the following pairs of lines are parallel :—

(i) $\begin{cases} 4x + 2y = 5. \\ 12x + 6y + 7 = 0. \end{cases}$
(ii) $\begin{cases} 3x - 2y = 4. \\ 6x - 4y = 3. \end{cases}$
(iii) $\begin{cases} 3x - y = 4. \\ 15x - 5y = 2. \end{cases}$
(iv) $\begin{cases} ax + by = 0. \\ kax + kby + d = 0. \end{cases}$

6. Prove that the following pairs of lines are perpendicular :—

(i) $\begin{cases} 3x - y = 4. \\ 5x + 15y = 2. \end{cases}$
(ii) $\begin{cases} 6x + 4y = 7. \\ 28x - 42y = 9. \end{cases}$
(iii) $\begin{cases} 2x - 3y = 1. \\ 15x + 10y = 4. \end{cases}$
(iv) $\begin{cases} x \cos \alpha + y \sin \alpha = p_1. \\ x \sin \alpha - y \cos \alpha = p_2. \end{cases}$

7. Tell which of the following lines pass through the origin and which are parallel to an axis :—

(i) $2x + 3y = 0$.
(ii) $y = 5$.
(iii) $x = -6$.
(iv) $x = 2y$.
(v) $y = -\frac{6}{5}$.
(vi) $ax - by = 0$.
(vii) $10x = 17$.
(viii) $x \cos \alpha + y \sin \alpha = 0$.

§§ (VIII-XIV). 8. Find the points of intersection of the following pairs of lines :—

(i) $3x - 5y + 1 = 0$ and $4x - 3y = 6$.
(ii) $4x + 5y = 3$ and $2x - 9y = 13$.
(iii) $6x = 5y - 8$ and $4x - 7y = 2$.

9. Give the gradients of the following lines, and say where they cross the y-axis :—

(i) $y = 4x + 8$.
(ii) $y = \frac{2}{3}x - 6$.
(iii) $y = -\frac{7}{5}x + 4$.
(iv) $y = 2x$.

10. Put into gradient form the equations which follow :—

(i) $3x - 2y = 1$.
(ii) $x = 5y - 5$.
(iii) $7x - 1 = 4y$.
(iv) $3x + 5y + 7 = 0$.
(v) $7 - 3y = -6x$.
(vi) $lx + my + n = 0$.

11. From the data below derive, in gradient form, the equation of each corresponding line :—

	gradient.	y-intercept.
(i)	$\frac{1}{2}$	3 positive
(ii)	3	4 negative
(iii)	$-\frac{3}{4}$	6 positive
(iv)	$-\frac{5}{2}$	8 negative

12. Throw the following equations into intercept form :—

$4x + 2y = 9$; $5x - 6y + 3 = 0$; $2x = 3y + 5$; $y = 2x - 4$.

13. Give the equation $3x - 5y + 30 = 0$ in intercept form.

14. Write in perpendicular form the equations of the lines derived from the data below :—

Length of the perpendicular from O to the line.	Angle between the perpendicular and OX.
(i) 4 units.	52°.
(ii) 5 units.	100°.
(iii) 6 units.	220°.
(iv) 8 units.	300°.

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§§ (XV-XX). 15. Find the equations of the lines which join the following pairs of points :—

(i) (1, 2) and (4, 5).
(ii) (− 2, 3) and (4, 6).
(iii) (− 1, − 2) and (9, 2).
(iv) (− 5, − 2) and (1, − 1).
(v) (− 4, 0) and (3, − 2).
(vi) (h, k) and $(0, b)$.

16. Find the equations of the lines which pass through the points named and have gradients as given :—

(i) (4, 1) ; gradient $\frac{3}{2}$.
(ii) (− 3, 2) ; gradient $-\frac{4}{5}$.
(iii) (5, − 4) ; gradient 2.
(iv) (− 6, − 8) ; gradient $-\frac{1}{3}$.

17. Show that the following sets of three points are collinear :—

(i) (− 1, − 2), (3, 2) and (4, 3).
(ii) (− 3, − 1), (0, 4) and (− 6, − 6).
(iii) (− 5, 3), (2, 0) and (9, − 3).

18. Show that the three lines in each of the following examples are concurrent :—

(i) $x - y + 5 = 0$; $2x + 3y = 0$; $5x + 6y + 3 = 0$.
(ii) $3x - 5y + 1 = 0$; $4x - 3y = 6$; $2x = 7y - 8$.

19. (*a*) Write down the equation of a system of lines through the intersection of the lines

$$x - 2y + 1 = 0 \text{ and } 3x - 8y + 9 = 0.$$

(*b*) Find the vertex of the pencil.

(*c*) Find the value of the parameter for the member which

(i) passes through the origin O.
(ii) has a gradient of $-\frac{4}{3}$.
(iii) is parallel to the line $5x + 2y = 12$.
(iv) is perpendicular to the member through O.

20. (*a*) Write down the equation of a system of lines passing through the intersection of $2x - 5y = 3$ and $3x + 4y = 16$.

(*b*) Find the vertex of the pencil.

(*c*) Which member of the system

(i) passes through O ?
(ii) makes a negative intercept of unit length on the x-axis ?
(iii) has a gradient of $\frac{1}{2}$?
(iv) is parallel to the y-axis ?

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Harder Examples

21. Find the equation of the straight line which passes through the point (3, 1) and makes a negative intercept of 4 units on the y-axis.

22. Prove that the lines joining the points [(2, 1), (4, 4)] and [(5, − 4), (− 1, 0)] are perpendicular.

23. By factorising the left-hand member of each of the following equations, show that the relative graph consists of a pair of straight lines. Say through what point all the lines pass :—

(i) $3x^2 - 5xy - 2y^2 = 0$. (iii) $4x^2 - 11xy + 6y^2 = 0$.
(ii) $8x^2 - 22xy + 15y^2 = 0$. (iv) $2x^2 - 7xy + 3y^2 = 0$.

24. Show that the graph of $10x^2 - 21xy - 10y^2 = 0$ consists of two perpendicular lines through O.

25. Solve for y the equation $2x^2 - 6xy + 3y^2 = 0$. Hence find the gradients and the inclinations of each of the two straight lines which form the graph.

Deduce the size of the acute angle between the lines.

26. **Prove that $lx + my + p = 0$ and $lx + my + q = 0$ are parallel lines, and that $lx + my + n = 0$ and $mx - ly + r = 0$ are perpendicular lines.**

27. Find the equations of the lines which are 2 units away from line (i) and 3 units away from line (ii) of example 14.

28. A point $P(x, y)$ moves so that its distance from the x-axis is one-third of its distance from the y-axis. Deduce the locus of the point, from the equation between x and y.

29. Perpendiculars PN and PM are drawn from a variable point $P(x, y)$ to the axes. If the length of the rectangle $ONPM$ exceeds the breadth by 2 units, deduce the locus of P from the equation connecting x and y.

30. Perpendiculars are drawn from a variable point to the axes forming a rectangle, the perimeter of which is 18 units. Find the locus of the point.

31. Find the gradients of the bisectors of the angles XOY and YOX'.

32. The straight line $2x + 5y = 10$ cut the axes at A and B. Find the co-ordinates of C, the mid-point of AB. Find also the equation of OC.

33. Perpendiculars PN and PM are drawn from the point (6, 4) to $X'X$ and $Y'Y$ respectively. N is joined to L the mid-point of PM. Find the equation of NL. Find also where NL cuts OY.

34. In the rectangle $OACB$, $A \equiv (a, 0)$ and $B \equiv (0, b)$. Find the equation of the line joining B to D, the mid-point of AC. Find also where the line crosses OX.

35. $OACB$ is a square such that $C \equiv (c, c)$. Find the equation of AB.

36. Points $A \equiv (1, 0)$, $B \equiv (0, 4)$ and $H \equiv (2, 3)$ are taken.

(i) Find P and Q the points where AH and BH respectively cut the axes again.

(ii) Deduce the equation of PQ.

37. $OACB$ is a square such that $C \equiv (1, 1)$. Find the equation of the line which passes through the intersection of the diagonals and the point (3, 0). Where does the line cut $Y'Y$?

38. Find the mid-points of the sides of the triangle whose sides are $x + 2y = 4$; $2x + y = 5$; $4x + 5y = 1$.

39. Find the median centre of the triangle

(i) with vertices (0, 0), (9, 4), (0, 2).

(ii) with sides $x = 2y$; $7x = 2y$; $5x + 2y = 24$.

(iii) with sides $8x^2 - 26xy + 15y^2 = 0$; $x + y = 7$.

40. Find the ortho-centre of the triangle

(i) with vertices (0, 0), (3, 3), (0, 4).

(ii) with sides $x = y$; $x = 2y$; $x + 2y = 8$.

(iii) with sides $x - 2y + 4 = 0$; $x + y = -1$; $x + 7y = 23$.

(iv) with sides $x^2 - 3xy + 2y^2 = 0$; $x + 2y = 3$.

[Hint. Find the equations of two of the altitudes.]

41. Find the equation of the line

(i) which passes through the point (3, 4) and is parallel to

$$x + 2y = 5 ;$$

(ii) which passes through the point (2, − 3) and is parallel to

$$4x - 5y = 2.$$

42. Find the equation of (i) the straight line which passes through the point (1, 1) and is perpendicular to $5x - 7y = 2$; (ii) the straight line which passes through the origin, and is perpendicular to $7x + 9y = 2$.

43. Find the equations of the straight lines through the origin which are (i) parallel, (ii) perpendicular to the following lines : $y = mx + b$; $\frac{x}{a} + \frac{y}{b} = 1$; $x \cos \alpha + y \sin \alpha = p$.

44. Obtain the equations of the diagonals of the square $OACB$ where $C \equiv (c, c)$. Show from the equations that the diagonals are perpendicular.

45. Find the equation of the line drawn through the intersection of $2x + y + 1 = 0$ and $6x - 5y = 37$, parallel to $2x - 3y = 0$.

46. Perpendiculars are drawn from the points $A(6, 0)$ and $B(0, 4)$ to the bisectors of the angles XOY and YOX'. Find the vertices of the rectangle thus formed, and prove that its area is half that of the rectangle contained by OA and OB.

47. In the square $OACB$, F is the mid-point of OB and E is taken on OC, so that $OE : EC = 1 : 2$. Prove that BE is perpendicular to FC.

48. $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and $P \equiv (x, y)$. If PA is perpendicular to PB, prove $x^2 + y^2 = a^2$.

49. Find the co-ordinates of the foot of the perpendicular from the origin to the lines (i) $2x + y = 10$; (ii) $y = mx + b$.

50. Through the point $D(h, k)$ a straight line is drawn perpendicular to OD. Find the equation of the line and the intercepts it makes on the axes.

51. Find the equations of the medians of the triangles whose vertices are (i) $[(2, 13), (4, 3), (12, 5)]$; and (ii) $[(-7, -1), (-1, 3), (5, -1)]$. In each case show that the medians are concurrent.

52. Find the equations of the medians of the triangle whose vertices are $(0, 0)$, $(a, 0)$ and $(0, b)$. Thence show that the medians are concurrent.

53. In the square $OACB$, M is the mid-point of AC and N is the point of trisection on the diagonal OC, nearer to C. If $C \equiv (6, 6)$, find the co-ordinates of N and prove B, N and M collinear.

54. The straight line $2x + y = 4$ cuts the axes at A and B. A triangle PQR is drawn such that the equation of PQ is $y = -3$, the equation of QR is $2x + y = 12$ and the equation of RP is $x = -1$.

Prove that PO, QA and RB are concurrent.

55. Prove that the line through the point $\left(-3, \frac{6}{t}\right)$ perpendicular to $t^2x - ty + 3 = 0$ cuts $X'X$ at the same point for all values of t.

56. If the three lines $x + 2y = 9$, $3x - 5y = 5$ and $\alpha x + \beta y = 1$ are concurrent, prove that the point (α, β) lies on the line $5x + 2y = 1$.

57. Obtain the orthocentre of the triangle whose vertices are $(-2, 5)$, $(2, -1)$ and $(4, 1)$.

58. From $C(6, 4)$ perpendiculars CA and CB are drawn to $X'X$ and $Y'Y$ respectively. D is the point $(5, 1)$. LDN is drawn parallel to OA to meet OB at L and AC at N. KDN is drawn parallel to OB to meet OA at K and BC at M. Prove BN, AM and OD concurrent.

59. In the figure of example 66, prove also that the following sets of lines are concurrent :—

(i) (LC, OM, AD); (ii) (LK, MN, DB).

60. $A \equiv (6, 0)$, $B \equiv (0, 2)$ and $C \equiv (-3, 3)$. Through the point C is drawn a straight line of gradient 3 to meet the x-axis at P and the y-axis at Q. Show that PB and AQ intersect on the bisector of $\widehat{XOY}$.

61. P is the point $(5, 5)$. PN is perpendicular to $X'X$. If $H \equiv (1, 3)$ and NH cuts $Y'Y$ at L, prove that LP passes through the point $(1, 4)$.

62. $B \equiv (0, 8)$, $H \equiv (3, 2)$ and $K \equiv (3, 5)$. BH meets $X'X$ in N, and the parallel to $Y'Y$ through N meets the bisector of $\widehat{XOY}$ in P. Prove B, K and P collinear.

63. Find the locus of the point $(x, y) \equiv (2t - 1, 5t + 2)$.

64. If the line $7x = 4y - 1$ passes through $P(5t - 4, t + 1)$, find the co-ordinates of P.

65. Find the co-ordinates of $P(3t^2, 6t)$, a point on the line $2y = x + 9$.

66. Prove that for all values of t the point

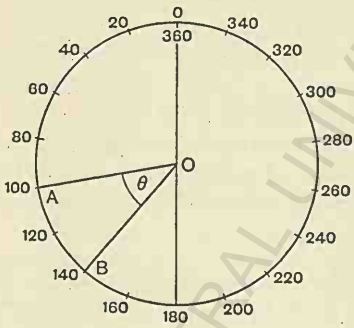
$$\left(mt - \frac{n}{2l}, -lt - \frac{n}{2m}\right)$$

lies on the line $lx + my + n = 0$.

CHAPTER III

ANGLE BETWEEN TWO LINES: PERPENDICULAR FROM A POINT TO A LINE: BISECTORS OF THE ANGLES BETWEEN TWO LINES

I. *Preliminary. The measurement and sign of an angle.* A brief reminder dealing with the measurement and sign of an angle will first be given.



The diagram represents a dial the circumference of which is supposed to be divided into 360 equal parts. The divisions and the scale numbers are, however, shown only at intervals of 20 units.

Consider now the measurement of the angle θ. If we reckon from A towards B we reckon forwards, with the scale, and say the angle is positive and 40°. On the other hand, if we reckon from B towards A we reckon against the scale, and say the angle is negative and of size $-40°$. This is often expressed otherwise as follows, that if the angle is formed by rotating the line OA into the position OB—that is, by a counter-clockwise movement—the angle is to be regarded as positive, but if it is formed by rotating the line OB into the position OA—that is, by a clockwise movement—then it is to be regarded as negative.

The angle taken in the positive sense is named AOB; in the negative sense BOA.

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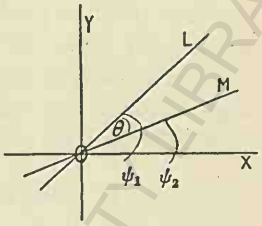
II. *Formula for the tangent of the angle between two given straight lines.*

Let ψ_1 and ψ_2 be the inclinations of the two given lines.

Draw OL parallel to the first and OM parallel to the second line.



Then $\widehat{MOL} = \theta$, the angle between the given lines.

(i) Let the gradients of the lines be m_1 and m_2. Then, since $\theta = \psi_1 - \psi_2$,

$$\therefore \tan\theta = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Let the equations of the lines be $l_1x + m_1y + n_1 = 0$ (inclination ψ_1), and $l_2x + m_2y + n_2 = 0$ (inclination ψ_2). Then $\tan\psi_1 = -\frac{l_1}{m_1}$, and $\tan\psi_2 = -\frac{l_2}{m_2}$. [We are here assuming that the inclination of the line (l_1, m_1, n_1) is greater than that of (l_2, m_2, n_2).]

$$\therefore \tan\theta = \frac{\tan\psi_1 - \tan\psi_2.}{1 + \tan\psi_1 \tan\psi_2}$$

$$= \frac{-\frac{l_1}{m_1} + \frac{l_2}{m_2}}{1 + \frac{l_1 l_2}{m_1 m_2}} = \frac{l_2 m_1 - l_1 m_2}{l_1 l_2 + m_1 m_2}.$$

Remarks. (*a*) In the demonstration the order of the lines implies that the included angle θ is formed by a counter-clockwise rotation, and is therefore positive. In this event, if the angle is acute, $\tan\theta$ will be positive, but if obtuse $\tan\theta$ will be negative. If, however, the order of

the lines be reversed in the formula, that is to say, if the angle be formed by a clockwise rotation, and therefore be negative, then $\tan\theta$ will be negative when the angle is acute and positive when the angle is obtuse [$\tan(-\theta) = -\tan\theta$].

(*b*) In numerical examples demanding the size of the angle between two straight lines, it is usual to give as answer the positive acute angle which the lines define.

EXAMPLE 1. *Find the angle between the lines* $6x - 4y = 3$ *and* $2x = 5y + 1$.

Writing the second equation as $-2x + 5y + 1 = 0$ we have $l_1 = 6$, $m_1 = -4$, $l_2 = -2$ and $m_2 = 5$,

$$\therefore \tan\theta = \frac{l_2m_1 - l_1m_2}{l_1l_2 + m_1m_2}$$

$$= \frac{(-2 \times -4) - (6 \times 5)}{(6 \times -2) + (-4 \times 5)}$$

$$= \frac{-22}{-32} = 0{\cdot}6875.$$

The acute angle between the lines is therefore 34° 21′.

Note that the first equation is that of the line of greater inclination, hence the rotation which defines θ is counter-clockwise or positive. If the second equation had been chosen to give l_1 and m_1 then the rotation would be clockwise or negative, and the formula would give $\tan\theta = -0{\cdot}6875$, as the reader can easily verify.

EXAMPLE 2. *Calculate the angle between the lines* $2x + 5y = 6$ *and* $6x + 15y + 7 = 0$. *Hence show that the lines are parallel.*

Putting $l_1 = 2$, $m_1 = 5$, $l_2 = 6$ and $m_2 = 15$, we have

$$\tan\theta = \frac{30 - 30}{12 + 75} = \frac{0}{87} = 0,$$

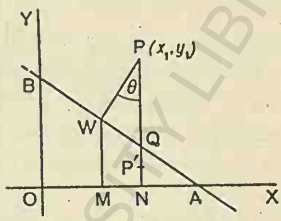
$$\therefore \theta = 0°.$$

The lines are therefore either coincident or parallel. They are clearly not coincident (since, for example, they cross the y-axis at different points), hence they are parallel.

III. *The expression $lx + my + n$ has the same sign for all points (x, y) on one side of $lx + my + n = 0$, and has the contrary sign for those on the other side of the line.*

Let AB be the line $lx + my + n = 0$.

Let $P(x_1, y_1)$ be any point in the plane of the axes.



Draw PN perpendicular to $X'X$ cutting AB at Q. Then $Q \equiv (x_1, NQ)$. Since Q lies on AB,

$$\therefore lx_1 + m \,.\, NQ + n = 0,$$

$$\therefore lx_1 + n = -m \,.\, NQ,$$

$$\therefore lx_1 + my_1 + n = my_1 - m \,.\, NQ$$

$$= m(NP - NQ).$$

Hence, since m is a constant, the sign of $lx_1 + my_1 + n$ depends on the difference $NP - NQ$ which is positive or negative according as NP is greater or less than NQ. Plainly NP is greater than NQ for points on the "upper" side of the line and less for points on the "lower" side of the line.

EXAMPLE 1. *Is the point* (2, 5) *on the same side of the line* $3x + 4y = 12$ *as the origin?*

First write the equation as, $-3x - 4y + 12 = 0$.
Then, when $x = 0$ and $y = 0$, $-3x - 4y + 12 = 12$,
and when $x = 2$ and $y = 5$, $-3x - 4y + 12 = -14$.

Since the results of these substitutions have contrary signs, the origin and (2, 4) lie on opposite sides of the line.

EXAMPLE 2. *Are the points* (1, 2) *and* (4, −3) *on the same or opposite sides of* $y = -3x$?

First write the equation as $3x + y = 0$.

Then when $x = 1$ and $y = 2$, $3x + y = 5$,
and when $x = 4$ and $y = -3$, $3x + y = 9$.

The points are therefore on the same side of the line.

Note. When the line does not pass through the origin, it is usual so to arrange the equation that, with all the

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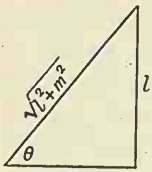
terms on one side, the absolute (or constant) term is positive. The expression $lx + my + n$ will then be positive for points on the origin side. When it passes through the origin, that is when the equation has the form $lx + my = 0$, the x-term is made positive. For example, if the equation were $3x = 2y$ we would write it $3x - 2y = 0$. In this event the expression is positive for points on the right-hand side of the line as can easily be verified. It is not necessary, however, to observe these conventions. We must simply be consistent throughout a given piece of work.

IV. *Length* (p) *of the perpendicular* PW *drawn from* $P(x_1, y_1)$ *to the line* $lx + my + n = 0$.



Use the figure of § III, where PW is drawn perpendicular to AB.

Then $\tan\theta = \cot P\widehat{Q}W$

$= \tan Q\widehat{A}N = -\tan\psi = \dfrac{l}{m}$.

Now the work of last article gives

$$lx_1 + my_1 + n = m(NP - NQ) = m \,.\, QP$$

$$= m \,.\, p \sec\theta$$

$$= mp \times \frac{\sqrt{l^2+m^2}}{m}$$

$$= p\sqrt{l^2+m^2},$$

$$\therefore\ p = \frac{lx_1 + my_1 + n}{\sqrt{l^2+m^2}}.$$

Remarks. Since we take the positive value of $\sqrt{l^2+m^2}$, p will have a sign which follows that of $lx_1 + my_1 + n$, and, when the line $lx + my + n = 0$ does not pass through the origin, the perpendiculars drawn from the origin side will easily be distinguished from those which are drawn from the other side. If, as described in § III,

the equation of the line be arranged so as to have a positive absolute term, then the perpendiculars from the origin side will be positive.

EXAMPLE 1. *Find the length of the perpendicular from the point* (1, 2) *to the line* $3x + 4y = 6$.

Writing the equation of the line as $-3x - 4y + 6 = 0$, and putting $x_1 = 1$ and $y_1 = 2$ in the formula for p, we have

$$p = \frac{-(3 \times 1) - (4 \times 2) + 6}{\sqrt{9 + 16}} = -1.$$

The perpendicular is of unit length.

EXAMPLE 2. *Find the length of the perpendicular from the origin to the line of last example.*

Putting $x_1 = 0$ and $y_1 = 0$ in the formula for p, gives

$$p_0 = \frac{-(3 \times 0) - (4 \times 0) + 6}{\sqrt{9 + 16}} = 1{\cdot}2.$$

The perpendicular from the origin is therefore 1·2 units long.

Note. Comparison of the sign of p_0 with that of p in the previous example shows that (1, 2) and the origin are on opposite sides of the line.

EXAMPLE 3. *Find the locus of a point which moves at a distance of* 2 *units from* $3x + 4y = 12$ *and on the origin side of the line.*

Let (x_1, y_1) be some position of the variable point.

[Use the figure of next example, placing (x_1, y_1) at P and (x_2, y_2) at Q.]

Write the equation of the line as $-3x - 4y + 12 = 0$.

Then since the absolute term of the left-hand side of the equation is positive, perpendiculars from the origin side will be positive.

The perpendicular from (x_1, y_1) being 2 units long and drawn from the origin side of the line, we have $2 = \frac{-3x_1 - 4y_1 + 12}{5}$, whence $3x_1 + 4y_1 = 2$. This shows that (x_1, y_1) is always a point on the straight line $3x + 4y = 2$, which is therefore the required locus.

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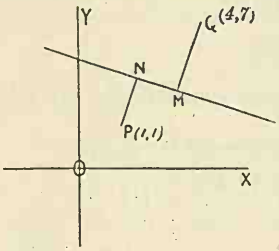
Note. If the variable point be on the side of the line away from O, as for example at $Q \equiv (x_2, y_2)$, then we have $\frac{-3x_2 - 4y_2 + 12}{5} = -2$, in which case the locus is $3x + 4y = 22$, a straight line parallel to $3x + 4y = 2$.

EXAMPLE 4. *A variable straight line is drawn so as to pass between the points* $P \equiv (1, 1)$ and $Q \equiv (4, 7)$. *If the ratio of the perpendiculars from* P *and* Q *to the line be always* $1 : 2$, *prove that it passes through a fixed point.*



Let the equation of the variable line be $y = mx + c$ . . (1)

$$\therefore PN = \frac{m - 1 + c}{\sqrt{m^2 + 1}} \text{ (numerically),}$$

and $$QM = \frac{4m - 7 + c}{\sqrt{m^2 + 1}}.$$

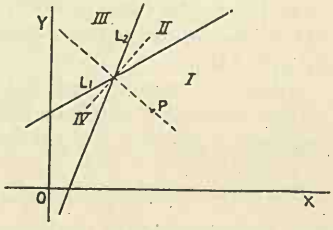
Now these two expressions must have contrary signs since P and Q lie on opposite sides of the line.

But $\frac{PN}{QM} = \frac{1}{2}$, $\therefore \frac{1}{2} = -\frac{m - 1 + c}{4m - 7 + c}$. Whence $c = 3 - 2m$, $\therefore y = mx + 3 - 2m$ [by substitution in (1)] or $y - 3 = m(x - 2)$.

Hence the variable line always passes through the fixed point (2, 3).

V. *Equations of the bisectors of the angles between two lines.*

Let L_1 and L_2 be the two lines, and let their equations be $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ respectively.



The lines divide the plane of the axes into four regions, numbered I, II, III and IV in the figure.

As it is a well-known geometrical property of the bisector of an angle that any point P on it is equidistant from the arms, we shall use this fact to obtain the equations of the bisectors.

First of all let us suppose that the origin is on the positive side of each line. This is arranged for by writing the equation of the line either as $lx + my + n = 0$ or $-lx - my - n = 0$, according to which form will secure the desired end. In compartment IV, then, we have $p_1 = p_2$, and in compartment II, $-p_1 = -p_2$, so that if the equations

$$l_1x + m_1y + n_1 = 0 \text{ and } l_2x + m_2y + n_2 = 0$$

be taken as arranged to put the origin on the positive side of each line, and if (x', y') be a point on the bisector through compartments II and IV, we have

$$\frac{l_1x' + m_1y' + n_1}{\sqrt{l_1^2 + m_1^2}} = \frac{l_2x' + m_2y' + n_2}{\sqrt{l_2^2 + m_2^2}}.$$

In other words, (x', y') lies on the line

$$\frac{l_1x + m_1y + n_1}{\sqrt{l_1^2 + m_1^2}} = \frac{l_2x + m_2y + n_2}{\sqrt{l_2^2 + m_2^2}}.$$

On the other hand, points in compartments I and III lie on the origin side of one line and the non-origin side of the other. Hence $p_1 = -p_2$ and the equation of the relative bisector is easily shown to be

$$\frac{l_1x + m_1y + n_1}{\sqrt{l_1^2 + m_1^2}} = -\frac{l_2x + m_2y + n_2}{\sqrt{l_2^2 + m_2^2}}.$$

EXAMPLE. *Obtain the equations of the bisectors of the angles between the lines* $3x - 4y = 7$ *and* $12x + 5y = 15$. *Deduce that the bisectors are at right angles to each other.*

To put the origin on the positive side of each line write

$$-3x + 4y + 7 = 0 \text{ and } -12x - 5y + 15 = 0.$$

The equations of the bisectors are therefore

$$\frac{-3x + 4y + 7}{5} = \frac{-12x - 5y + 15}{13},$$

and

$$\frac{-3x + 4y + 7}{5} = -\frac{-12x - 5y + 15}{13}.$$

Whence $21x + 77y + 16 = 0$ and $99x - 27y - 166 = 0$.

Now the product of the gradients is $-\frac{21}{77} \times \frac{99}{27} = -1$, showing the lines to be at right angles.

EXAMPLES III

§ (II). 1. Find the acute angle between the lines whose gradients are (i) $\frac{1}{2}$ and $\frac{3}{4}$, (ii) $-\frac{2}{5}$ and $\frac{1}{3}$, (iii) $-\frac{3}{4}$ and $-\frac{5}{6}$.

2. Find the acute angle between the lines $3x - 2y = 1$ and $4x + y = 2$, (i) by using the formula of § II, and (ii) by finding the inclination of each line and calculating the difference.

3. Find the acute angle between the following pairs of lines :—

(i) $2x - 3y + 12 = 0$.
$x - 5y + 5 = 0$.

(ii) $4x + 6y - 5 = 0$.
$3x + 5y + 1 = 0$.

(iii) $7x - 4y = 1$.
$5x + 6y = 3$.

(iv) $4x + 5y = 3$.
$3x - 2y + 4 = 0$.

(v) $3x - 2y = 4$.
$6x - 4y = 3$.

(vi) $2x - 3y = 1$.
$6x + 4y = 7$.

Tell whether any pair is parallel or perpendicular.

§§ (III-IV). 4. Are the points $(2, -5)$ and $(-3, 4)$ on the same or opposite sides of the line $4x - 6y = 7$?

5. Classify the following points relative to the line
$3x = 2y - 5$, $(1, 0)$, $(2, 7)$, $(-4, -1)$, $(3, -5)$, $(0, 8)$.

6. Classify the following points relative to the line
$2x + 5y = 0$; $(-3, 1)$, $(-1, 4)$, $(0, -6)$, $(2, 3)$.

7. Find the length of the perpendicular from the point to the line in each of the following cases :—

(i) $(2, 1)$; $4x + 3y + 5 = 0$.
(ii) $(2, 3)$; $5x + 12y = 20$.
(iii) $(-4, 5)$; $3x = 4y - 7$.
(iv) $(-3, -4)$; $5y = 12x + 3$.
(v) $(0, 0)$; previous lines.
(vi) $(5, -3)$; $8x = 15y$.
(vii) $(0, 0)$; $y = mx + b$.
(viii) $(0, 0)$; $\frac{x}{a} + \frac{y}{b} = 1$.
(ix) $(\tan \alpha, \cot \alpha)$; $x \cos \alpha + y \sin \alpha = p$.
(x) $(m, -1)$; $y = mx$.
(xi) $(m, b - 1)$; $y = mx + b$.
(xii) $(l-m, m-l)$; $lx + my = 0$.

§ (V). 8. Find the equations of the bisectors of the angles between the following pairs of lines :—

(i) $x = 2y$. $2x = y$.
(ii) $3x + 4y = 7$. $5x = 12y + 21$.
(iii) $8x - 15y + 14 = 0$. $3y + 2 = 4x$.
(iv) $5x - 12y + 5 = 0$. $8x + 15y + 8 = 0$.
(v) $2x^2 + 5xy - 12y^2 = 0$.
(vi) $x + y = 0$. $x - y = 0$.

Harder Examples

9. Find the length of the perpendicular from the point (5, 2) to the line $2x - 3y = 4$. What do you conclude ?

10. From the point $C(4, 6)$ perpendiculars CA and CB are drawn to $X'X$ and $Y'Y$ respectively. If D is the mid-point of AC find the length of the perpendicular from B to OD.

11. Through $B'(0, -4)$ is drawn a line of gradient $\frac{1}{2}$. To this line is drawn from $B(0, 4)$ a perpendicular BD. Calculate the lengths of BD and $B'D$, and so find the area of $\triangle BDB'$. Verify the previous result by finding the co-ordinates of D.

12. Find the length of the perpendicular from the point (a, b) to the line $\frac{x}{a} + \frac{y}{b} = 1$.

13. Show that the origin and the points $(\sin\alpha, -\cos\alpha)$ and $(-\sin\alpha, \cos\alpha)$ are equidistant from $x\cos\alpha + y\sin\alpha = p$ and lie on the same side of the line.

14. Find the locus of a point which moves on the origin side at a distance of 3 units from the line $4x + 3y = 5$.

15. A point is taken 5 units from the line $3x + 4y = 6$. Find the equations of the two lines on which it may lie.

16. $x\cos\alpha + y\sin\alpha = p$ is drawn so that the points $P(-1, -1)$ and $Q(5, 8)$ lie on opposite sides of the line. If the ratio of the perpendiculars from P and Q to the line be 1 : 2, prove that it must pass through the point (1, 2).

17. The points $P(2, 3)$ and $Q(1, 4)$ lie on the same side of the variable line $x\cos\alpha + y\sin\alpha = p$. If 4 : 3 be the ratio of the perpendiculars from P and Q to the line, show that it passes through a fixed point.

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18. Find the distance between the lines

(i) $3x + 4y = 12$ and $3x + 4y = 2$.
(ii) $12x - 5y = 18$ and $12x - 5y + 8 = 0$.
(iii) $x \cos \alpha + y \sin \alpha = p_1$ and $x \cos \alpha + y \sin \alpha = p_2$,
(iv) $y = mx + b$ and $y = mx + c$.

19. By calculating the angles between the sides of the triangle whose vertices are (0, 2), (2, 0) and $(1 + \sqrt{3}, 1 + \sqrt{3})$, show that the triangle is equilateral.

20. $OACB$ is a rectangle such that $A \equiv (3, 0)$ and $B \equiv (0, 1)$. From B a perpendicular is drawn to OC. From A perpendiculars are drawn to this line and to OC. Find the area of the rectangle so formed.

21. In triangle OAB, $\widehat{AOB} = 90°$, $A \equiv (a, 0)$ and $B \equiv (0, b)$. Obtain the equations of the bisectors of angles OAB and OBA.

Thence show that these lines intersect on the bisector of $\widehat{AOB}$.

22. A circle is drawn with O as centre to touch the line $8x + 15y = 34$. Find the length of the radius.

23. A circle is drawn with centre $(-3, 1)$ to touch the line $12x - 5y - 24 = 0$. Find the length of the radius.

24. The line $3x + 4y = k$ is drawn to touch the circle with centre O and radius 12 units. What two values may k have ?

25. One side of a square circumscribing the circle with centre (2, 1) and radius of 4 units is the line $3x + 4y + n = 0$. Find the equations of the sides.

26. In the rectangle $OACB$, $A \equiv (8, 0)$, $B \equiv (0, 6)$ and $E \equiv (8, y_1)$. Another rectangle is drawn on OE with the side opposite OE passing through B. Find expressions for the dimensions of this rectangle and so prove that its area is constant for all values of y_1.

27. By calculating the length of the perpendicular from the origin to the line $2x - 3y + 4 = 0$, verify that the "Perpendicular" form of the equation is $-\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y = \frac{4}{\sqrt{13}}$.

Use the same procedure to turn the following equations into perpendicular form :—

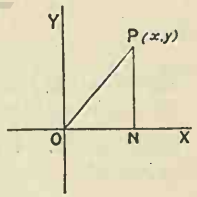
$$x + 4y = 23\,;\ 5x = 2y - 18\,;\ 6x + 8y + 25 = 0.$$

CHAPTER IV

DISTANCE BETWEEN TWO POINTS: EQUATIONS OF CIRCLE, CHORD, TANGENT: CHORD OF CONTACT: LENGTH OF TANGENT

I. *Distance of the point $P(x, y)$ from the origin.*

Draw PN perpendicular to the x-axis. Join OP and let $OP = d_0$.



Then $OP^2 = ON^2 + NP^2$,

$$\therefore\ d_0^2 = x^2 + y^2,$$

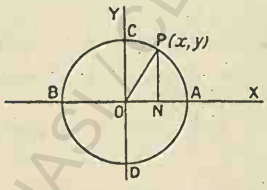
$$\therefore\ d_0 = \sqrt{x^2 + y^2}.$$

EXAMPLE. *Find the distance of the point* $(-3, -4)$ *from the origin.*

We have $d_0^2 = x^2 + y^2 = (-3)^2 + (-4)^2 = 25,$

$$\therefore\ d_0 = 5.$$

II. *Equation of the circle with centre O and radius of a units.*



Let $P(x, y)$ be any point on the circumference of the circle.

Then, as in Section I,

$$x^2 + y^2 = a^2.$$

This equation is called the equation of the circle having O as centre and radius a units long, as it is the condition which the co-ordinates of points on the circumference must satisfy.

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EXAMPLE 1. *Write down the equation of the circle which has O as centre and a radius 6 units long.*

In this case $a = 6$, in the equation $x^2 + y^2 = a^2$.

The required equation is therefore $x^2 + y^2 = 36$.

EXAMPLE 2. *Where does the straight line $x - 2y + 5 = 0$ cut the circle $x^2 + y^2 = 10$?*

In Chapter I, Section II (*e*), we saw that the points of intersection of two graphs could be obtained by solving the relative equations.

We have $x^2 + y^2 = 10 \quad . \quad . \quad . \quad . \quad (1)$

and $x - 2y + 5 = 0 \quad . \quad . \quad . \quad . \quad (2)$

Equation (2) gives $x = 2y - 5$.

Substitute for x in equation (1),

$$\therefore (2y - 5)^2 + y^2 = 10,$$

whence $$y^2 - 4y + 3 = 0,$$

$$\therefore y = 1 \text{ or } 3.$$

Equation (2) now gives $x = -3$ or 1.

The points of intersection are $(1, 3)$ and $(-3, 1)$.

EXAMPLE 3. *Where does the straight line $y = 3x + 10$ meet the circle $x^2 + y^2 = 10$?*

$$x^2 + y^2 = 10 \quad . \quad . \quad . \quad . \quad (1)$$

$$y = 3x + 10 \quad . \quad . \quad . \quad . \quad (2)$$

Substituting for y from (2) in (1) we have

$$x^2 + (3x + 10)^2 = 10,$$

whence $$x^2 + 6x + 9 = 0,$$

$$\therefore (x + 3)(x + 3) = 0,$$

$$\therefore \text{ both values of } x \text{ are } -3.$$

Equation (2) now gives both values of y as 1. The straight line therefore cuts the circle in two points coincident with $(-3, 1)$.

In other words, the straight line is the tangent at $(-3, 1)$.

EXAMPLE 4. *In what ratio does the circle $x^2 + y^2 = 25$ divide the join of the points $A(5, 5)$ and $B(1, 3)$?*

Let the circle divide the straight line AB at a point (x_1, y_1) in the ratio $k : 1$,

$$\therefore x_1 = \frac{k + 5}{k + 1}$$

and

$$y_1 = \frac{3k + 5}{k + 1}.$$

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Now (x_1, y_1) is a point on the circle,

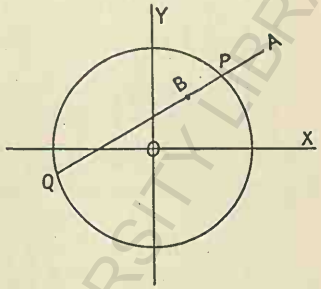
$$\therefore\ x_1^2 + y_1^2 = 25, \quad \therefore \left(\frac{k+5}{k+1}\right)^2 + \left(\frac{3k+5}{k+1}\right)^2 = 25,$$

$$\therefore\ (k+5)^2 + (3k+5)^2 = 25(k+1)^2,$$

whence

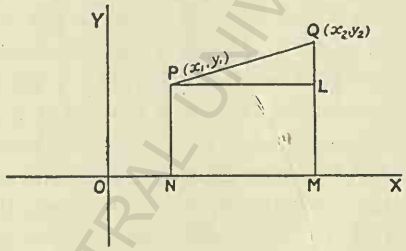
$$3k^2 + 2k - 5 = 0,$$

$$\therefore\ k = 1 \text{ or } -\tfrac{5}{3}.$$



The line AB is therefore cut internally at P, the point of bisection, and externally at Q in the ratio 5 : 3.

III. *Distance of the point* $P(x_1, y_1)$ *from the point* $Q(x_2, y_2)$.



Plot the points P and Q, and draw PN and QM perpendicular to the x-axis. Draw also PL parallel thereto.

Then $PQ^2 = PL^2 + LQ^2$.

Now $PL = NM = OM - ON = x_2 - x_1$,

and $LQ = MQ - ML = MQ - NP = y_2 - y_1$,

$$\therefore\ PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Let $PQ = d$, $\therefore\ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

EXAMPLE. *Find the length of the line joining the points* $(-1, 1)$ *and* $(3, -2)$.

$$d^2 = (3+1)^2 + (-2-1)^2 = 25, \quad \therefore\ d = 5.$$

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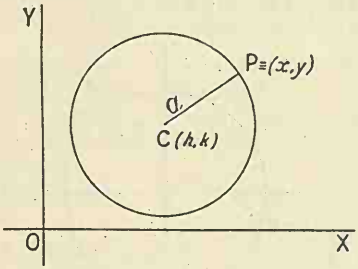
IV. *The equation of the circle with centre $C(h, k)$ and radius of a units.*



Let $P \equiv (x, y)$ be any point on the circumference of the circle. Then, since $CP^2 = a^2$, we have

$$(x - h)^2 + (y - k)^2 = a^2$$

which is therefore the required equation.

Remarks. On removing brackets from the equation $(x - h)^2 + (y - k)^2 = a^2$ and transposing terms we have $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0$.

Putting $h = -g$, $k = -f$ and $h^2 + k^2 - a^2 = c$, the equation then becomes $x^2 + y^2 + 2gx + 2fy + c = 0$.

Now $h^2 + k^2 = OC^2$. Hence when O is inside the circle and OC is therefore shorter than the radius, we have $h^2 + k^2 < a^2$, in which event $h^2 + k^2 - a^2$ or c is negative.

On the other hand, when O is outside the circle OC is greater than the radius making $h^2 + k^2 - a^2$ or c positive. In this case the student will remember that, if OT is the tangent from O to the circle, then $OT^2 = OC^2 - CT^2$, whence we have $c = OT^2$. (It is left to the reader to draw a figure if he finds one necessary.)

EXAMPLE 1. *Find the equation of the circle with* $(-2, 5)$ *as centre and a radius of* 3 *units.*

Substitution in the equation of a circle gives

$$(x + 2)^2 + (y - 5)^2 = 9,$$

whence $$x^2 + y^2 + 4x - 10y + 20 = 0.$$

EXAMPLE 2. *In example* 1, *is the origin inside or outside the circle?*

Since the absolute term of the equation is $+20$, the origin is outside the circle.

V. *The graph of the equation* $x^2 + y^2 + 2gx + 2fy + c = 0$ *is a circle.*

We have $(x^2 + 2gx) + (y^2 + 2fy) = -c$,

$\therefore (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$,

$\therefore (x + g)^2 + (y + f)^2 = g^2 + f^2 - c = \text{constant}$.

It follows that the distance between the variable point (x, y) and the fixed point $(-g, -f)$ is constant.

The locus of (x, y)—that is, the graph of the equation $x^2 + y^2 + 2gx + 2fy + c = 0$—is therefore a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Note that the co-ordinates of the centre are half the coefficients of x and y with the sign in each case reversed.

Note also that $kx^2 + ky^2 + 2gx + 2fy + c = 0$ is the equation of a circle, for we can write it as

$$x^2 + y^2 + \frac{2g}{k}x + \frac{2f}{k}y + \frac{c}{k} = 0.$$

It remains to remark that in the equation of a circle the coefficients of x^2 and y^2 are equal in magnitude and sign and there is no term in xy.

EXAMPLE 1. *Find the centre and the radius of the circle* $x^2 + y^2 + 6x - 4y - 3 = 0$.

We have, centre of circle $\equiv (-g, -f) \equiv (-3, 2)$

and radius of circle $= \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4 + 3} = 4$.

EXAMPLE 2. *Find where the straight line* $y = 3x - 1$ *cuts the circle* $x^2 + y^2 + 8x - 14y + 15 = 0$.

To find the points of intersection we must solve the equations of the graphs. Put $y = 3x - 1$ in the equation of the circle,

$$\therefore x^2 + (3x - 1)^2 + 8x - 14(3x - 1) + 15 = 0,$$

whence $x^2 - 4x + 3 = 0$, giving $x = 1$ or 3.

From the equation $y = 3x - 1$ we now obtain $y = 2$ or 8 as the corresponding values of y. Thus the points of intersection are (1, 2) and (3, 8).

EXAMPLE 3. *Show that the straight line* $5x + 2y = 18$ *touches the circle* $x^2 + y^2 + x - 6y + 2 = 0$.

Since $5x + 2y = 18, \quad \therefore y = \dfrac{18 - 5x}{2},$

$$\therefore x^2 + \frac{(18 - 5x)^2}{4} + x - \frac{6(18 - 5x)}{2} + 2 = 0,$$

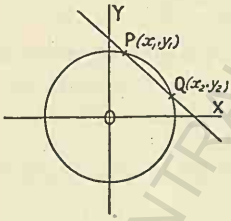
whence $x^2 - 4x + 4 = 0$; that is $(x - 2)^2 = 0$.

Thus both values of x are 2.

Now when $x = 2, \quad y = \dfrac{18 - 5x}{2} = 4.$

The line therefore cuts the circle in two points coincident with (2, 4). In other words, it is a tangent to the circle.

VI. *The chord joining the points* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *on the circle* $x^2 + y^2 = a^2$; *the tangent at* $P(x_1, y_1)$.



Since P and Q lie on the circle,

$$\therefore x_1^2 + y_1^2 = a^2 = x_2^2 + y_2^2,$$

$$\therefore y_1^2 - y_2^2 = -x_1^2 + x_2^2,$$

$$\therefore (y_1 - y_2)(y_1 + y_2) = -(x_1 - x_2)(x_1 + x_2),$$

$$\therefore \text{gradient of } PQ = \frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2}{y_1 + y_2}.$$

Corollary 1. If (x, y) be any third point on PQ, then

$$\frac{y - y_1}{x - x_1} = \text{gradient of } PQ = -\frac{x_1 + x_2}{y_1 + y_2}.$$

Hence the equation of the chord may be written

$$(x - x_1)(x_1 + x_2) + (y - y_1)(y_1 + y_2) = 0.$$

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Corollary 2. **(The case of the tangent at P is very important.)**

For the tangent at P, Q coincides with P. Hence $x_2 = x_1$ and $y_2 = y_1$, so that the equation of Corollary 1 becomes $2x_1(x - x_1) + 2y_1(y - y_1) = 0$. Simplify and rearrange. $\therefore\ xx_1 + yy_1 = x_1^2 + y_1^2$,

giving $xx_1 + yy_1 = a^2$ as the equation of the tangent at P.

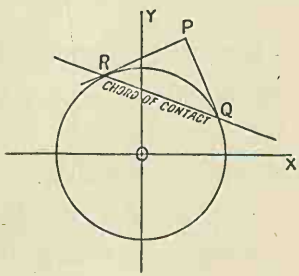
Corollary 3. Last equation shows that the gradient of the tangent at P is $-\frac{x_1}{y_1}$. But the gradient of OP is $\frac{y_1}{x_1}$, $\therefore$ gradient of tangent $\times$ gradient of $PQ = -1$, showing that the tangent at P is perpendicular to the radius OP.

EXAMPLE. *Show that the point* $(-2, 3)$ *lies on the circle* $x^2 + y^2 = 13$, *and then find the equation of the tangent at the point.*

When $x = -2$ and $y = 3$ then $x^2 + y^2 = 4 + 9 = 13$. Hence $(-2, 3)$ is a point on the circle.

Put $x_1 = -2$, $y_1 = 3$ and $a^2 = 13$ in $xx_1 + yy_1 = a^2$, $\therefore\ -2x + 3y = 13$ or $2x - 3y + 13 = 0$ is the required equation.

VII. *To find the equation of the chord joining the points of contact of two tangents to the circle* $x^2 + y^2 = a^2$ *which intersect at the point* $P(x', y')$.



Let the points of contact be $Q \equiv (x_1, y_1)$ and $R \equiv (x_2, y_2)$. The equation of the tangent at Q is $xx_1 + yy_1 = a^2$. Since P lies on this tangent, therefore $x'x_1 + y'y_1 = a^2$. Hence (x_1, y_1) is a point on the line $x'x + y'y = a^2$. Similarly, (x_2, y_2) is a point on this line. The equation $xx' + yy' = a^2$ is therefore that of the line which passes through the points Q and R. QR is called the chord of contact of the tangents

from P to the circle. The similarity of its equation to that of a tangent is obvious, but one must note that the values x' and y' do not satisfy the equation of the circle, as the co-ordinates of the point of contact of a tangent do.

EXAMPLE 1. *Find the equation of the chord of contact of tangents to the circle* $x^2 + y^2 = 3$ *which intersect at the point* (4, 6).

$x' = 4$, $y' = 6$ and $a^2 = 3$, $\therefore\ 4x + 6y = 3$ is the required equation.

EXAMPLE 2. *Find the point of intersection of the tangents drawn at the extremities of the chord intercepted by the circle* $x^2 + y^2 = 45$ *on the line* $x + y = 9$.

Let (x', y') be the point of intersection of the tangents. Then the equation of the chord of contact is $xx' + yy' = 45$.

This line must be identical with $x + y = 9$.

Identifying corresponding intercepts on the axes gives

$$\frac{45}{x'} = 9 \text{ (x-intercepts) and } \frac{45}{y'} = 9 \text{ (y-intercepts),}$$

$$\therefore\ x' = 5 \text{ and } y' = 5, \quad \therefore\ (x', y') \equiv (5, 5).$$

VIII. *Equation of the tangent at the point* $P \equiv (x_1, y_1)$ *on the circle* $x^2 + y^2 + 2gx + 2fy + c = 0$,

$$\because \text{ gradient of radius } CP = \frac{y_1 + f}{x_1 + g}, \quad [\text{Centre } C \equiv (-g, -f)],$$

$$\therefore \text{ gradient of tangent at } P = -\frac{x_1 + g}{y_1 + f}. \quad (\S\ \text{VI, Cor. 3.})$$

Hence, if (x, y) is any point on the tangent, we have

$$\frac{y - y_1}{x - x_1} = -\frac{x_1 + g}{y_1 + f} \text{ (each side measuring the gradient),}$$

$$\therefore\ yy_1 - y_1^2 + fy - fy_1 = -xx_1 - gx + x_1^2 + gx_1,$$

$$\text{or } xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1.$$

But since (x_1, y_1) is a point on the circle,

$$\therefore\ 0 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

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Whence, by subtraction,

$$xx_1 + yy_1 + gx + fy = -gx_1 - fy_1 - c,$$

giving $(x_1 + g)x + (y_1 + f)y + gx_1 + fy_1 + c = 0$. (*a*)

or $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$. (*b*)

as the equation of the tangent at (x_1, y_1).

EXAMPLE. *Find the equation of the tangent at the point* $(2, -5)$ *on the circle* $x^2 + y^2 - 5x + y - 14 = 0$.

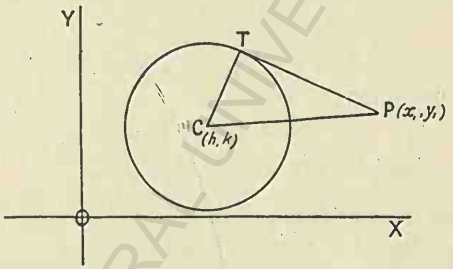
Here $g = -\frac{5}{2}$, $f = \frac{1}{2}$, $c = -14$, $x_1 = 2$ and $y_1 = -5$.

Substitution in (say) form (*a*) above gives

$$(2 - \tfrac{5}{2})x + (-5 + \tfrac{1}{2})y + (-\tfrac{5}{2} \times 2) + (\tfrac{1}{2} \times -5) - 14 = 0.$$

Whence $x + 9y + 43 = 0$ is the required equation.

IX. *Length of the tangent drawn from a fixed point* $P(x_1, y_1)$ *to the circle with centre* $C(h, k)$ *and radius of* a *units.*



$$\therefore PT^2 = CP^2 - CT^2, \quad \therefore PT^2 = (x_1 - h)^2 + (y_1 - k)^2 - a^2,$$

$$\therefore PT = \sqrt{(x_1 - h)^2 + (y_1 - k)^2 - a^2}.$$

Note. If the equation of the circle be written

$(x - h)^2 + (y - k)^2 - a^2 = 0$, *then if we put* $x = x_1$ *and* $y = y_1$ *in the left-hand side, the result is* PT^2.

Corollary. Where $x^2 + y^2 + 2gx + 2fy + c = 0$ is taken as the equation of the circle, then

$PT^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$. (See *Note* above.)

The reader should obtain the above result directly from the figure.

N.B. The equation of the circle must be arranged so as to have the coefficients of x^2 *and* y^2 *unity.*

EXAMPLE 1. *Find the length of the tangent drawn from the point* (10, 3) *to the circle* $x^2 + y^2 - 2x + 4y - 1 = 0$.

Now $PT^2 = 10^2 + 3^2 - 2 \cdot 10 + 4 \cdot 3 - 1 = 100$, $\therefore PT = 10$.

EXAMPLE 2. *Find the length of the tangent drawn from the point* (2, 1) *to the circle* $x^2 + y^2 - 2x + 4y - 12 = 0$.

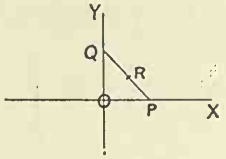
$PT^2 = 4 + 1 - 8 + 4 - 12 = -7$, $\therefore PT = \sqrt{-7}$.

The tangent is unreal; (2, 1) is inside the circle.

X. MISCELLANEOUS EXAMPLES

EXAMPLE 1. *A variable line of constant length c units cuts the axes at P and Q. Find the locus of R, the mid-point of PQ.*

Let $P \equiv (p, 0)$, $Q \equiv (0, q)$
and $R \equiv (x, y)$.



Then $x = \frac{p}{2}$ and $y = \frac{q}{2}$,

$\therefore p = 2x$ and $q = 2y$.

But $OP^2 + OQ^2 = PQ^2$,

$\therefore p^2 + q^2 = c^2$,

whence $4x^2 + 4y^2 = c^2$ or $x^2 + y^2 = \frac{c^2}{4}$.

Hence the locus of $R(x, y)$ is a circle whose centre is O and and whose radius is $\frac{1}{2}c$ units in length.

EXAMPLE 2. *Prove that two tangents can be drawn to the circle* $x^2 + y^2 = 5$ *through the point* (3, 1) *and find their equations.*

Let the equation of a tangent through (3, 1) be $y = mx + b$,

$$\therefore 1 = 3m + b \quad . \quad . \quad . \quad (1)$$

Now the circle $x^2 + y^2 = 5$ and the line $y = mx + b$ intersect where $x^2 + (mx + b)^2 = 5$,

Simplify, $\therefore (1 + m^2)x^2 + 2bmx + b^2 - 5 = 0$.

In the case of a tangent the roots of this equation are equal (see Chap. IV, § II, Ex. 3; and § VI, Cor. 2),

$\therefore 4b^2m^2 = 4(1 + m^2)(b^2 - 5)$, or $5m^2 - b^2 + 5 = 0$.

But from (1) above, we have $b = 1 - 3m$,

$\therefore 5m^2 - (1 - 3m)^2 + 5 = 0$, giving $2m^2 - 3m - 2 = 0$,

$\therefore m = 2$ or $-\frac{1}{2}$, whence $b = -5$ or $2\frac{1}{2}$.

The two tangents are $y = 2x - 5$ and $y = -\frac{1}{2}x + 2\frac{1}{2}$.

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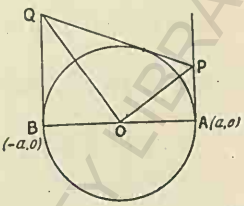
EXAMPLE 3. *AB is a diameter of a circle. Tangents are drawn at A and B. Any other tangent cuts these at P and Q. Prove that PQ subtends a right angle at the centre O of the circle.*



Take AB as axis of x and its right-bisector as axis of y. Let $A \equiv (a, 0)$. Then $B \equiv (-a, 0)$. Hence the equation of the circle is

$$x^2 + y^2 = a^2 \quad . \quad . \quad (1)$$

and of PQ the tangent at (x_1, y_1) is

$$xx_1 + yy_1 = a^2 \quad . \quad . \quad (2)$$

To find the co-ordinates of P put $x = a$ in equation (2),

$$\therefore ax_1 + yy_1 = a^2, \quad \therefore y = \frac{a^2 - ax_1}{y_1},$$

$$\therefore P \equiv \left(a, \frac{a^2 - ax_1}{y_1}\right).$$

Similarly, $$Q \equiv \left(-a, \frac{a^2 + ax_1}{y_1}\right).$$

The gradient of $OP = \dfrac{a^2 - ax_1}{ay_1} = \dfrac{a - x_1}{y_1}$.

The gradient of $OQ = \dfrac{a^2 + ax_1}{-ay_1} = -\dfrac{a + x_1}{y_1}$.

Hence the product of the gradients $= -\dfrac{a^2 - x_1^2}{y_1^2}$.

But $x_1^2 + y_1^2 = a^2$ and $\therefore y_1^2 = a^2 - x_1^2$.

The product of the gradients is therefore -1, $\therefore P\hat{O}Q = 90°$.

EXAMPLE 4. *Find the equation of the circle which passes through the three points* (6, 0), (− 4, 0) and (0, 8).

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the required equation. Then, since (6, 0), (− 4, 0) and (0, 8) lie on the circle,

$$\therefore 36 + 12g + c = 0 \quad . \quad . \quad . \quad (1)$$

$$16 - 8g + c = 0 \quad . \quad . \quad . \quad (2)$$

$$64 + 16f + c = 0 \quad . \quad . \quad . \quad (3)$$

Eqns. (1) and (2) give $g = -1$ and $c = -24$, $\therefore f = -\frac{5}{2}$,

$\therefore x^2 + y^2 - 2x - 5y - 24 = 0$ is the equation sought.

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EXAMPLE 5. *Find the equation of the circle on the join of* $A \equiv (x_1, y_1)$ *and* $B \equiv (x_2, y_2)$ *as diameter.*

Let $P \equiv (x, y)$ be any point on the locus.

Then the gradient of AP is $\dfrac{y - y_1}{x - x_1}$ and of BP is $\dfrac{y - y_2}{x - x_2}$.

Now $\widehat{APQ} = 90°$, $\therefore \dfrac{y - y_1}{x - x_1} \times \dfrac{y - y_2}{x - x_2} = -1$,

$$\therefore (y - y_1)(y - y_2) + (x - x_1)(x - x_2) = 0,$$

or $x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$.

EXAMPLES IV

§§ (I and II). 1. How far from the origin are (3, 4), (− 12, 5), (− 8, − 15), (2, 3), (− 4, 1), (− 2, − 5)?

2. Write down the equations of the circles having 0 as centre and radii of 1, 3 and 7 units respectively.

3. Give the length of the radius of each of the circles

(i) $x^2 + y^2 = 4$; (ii) $x^2 + y^2 = 9$; (iii) $x^2 + y^2 = 36$;
(iv) $4x^2 + 4y^2 = 25$; (v) $9x^2 + 9y^2 = 64$.

4. Find where the given straight lines cut the circles:—

(i) $y = 6 - x$; $x^2 + y^2 = 20$. (iii) $x + 3y = 8$; $x^2 + y^2 = 8$.
(ii) $y = 2x + 1$; $x^2 + y^2 = 10$. (iv) $3x - 4y = 1$; $x^2 + y^2 = 13$.

5. In the two examples below show that the straight line touches the circle and find the points of contact:—

(i) $2x - y = 5$; $x^2 + y^2 = 5$. (ii) $2x + 3y = 13$; $x^2 + y^2 = 13$.

6. In the examples below find the ratios in which the circles divide the joins of the pairs of points:—

(i) $x^2 + y^2 = 25$; (− 2, 3), (1, 6). (iv) $x^2 + y^2 = 13$; (1, 1), (7, 4).
(ii) $x^2 + y^2 = 20$; (1, 3), (− 5, 5). (v) $x^2 + y^2 = 13$; (1, 2), (3, 5).
(iii) $x^2 + y^2 = 10$; (− 3, − 5), (0, − 2).

§§ (III-V). 7. Find the distance between

(2, 4) and (6, 7); (3, 5) and (15, 10); (− 4, 2) and (8, − 14); (5, 3·5) and (3, 1·4); (− 3·6, − 8·3) and (− 1·2, − 3·8); (7, 0) and (− 9, − 12); (5c, − 4c) and (c, − c).

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8. Find the equations of the circles which have

Centre.	Radius.	Centre.	Radius.
(i) (4, 2)	3	(iv) (0, 3)	7
(ii) (− 6, 5)	2	(v) (− 2, 0)	4
(iii) (− 3, − 4)	5	(vi) (a, a)	a

9. Tell which of the following circles have the origin outside :—

(i) $x^2+y^2-6x+8y+10=0$. (iv) $x^2 + y^2 - 4x + 8 = 0$.
(ii) $x^2+y^2+5x-2y-6=0$. (v) $x^2 + y^2 + 7y - 4 = 0$.
(iii) $x^2+y^2-x-4y+7=0$. (vi) $5x^2 + 5y^2 - 3x + c^2 = 0$.

10. Give the centres and the radii of the circles—

$x^2 + y^2 - 8x - 6y + 21 = 0$; $x^2 + y^2 - 4x + 10y - 7 = 0$;
$x^2 + y^2 - 10x + 9 = 0$; $x^2 + y^2 + 24x - 10y = 0$;
$x^2 + y^2 = 7x + 6y - 1$; $x^2 + y^2 + x - 2y + 1 = 0$;
$x^2 + y^2 - 2x + 3y - 3 = 0$; $4x^2 + 4y^2 - 20x + 24y - 3 = 0$;
$16x^2 + 16y^2 + 16x - 24y = 87$; $4x^2 + 4y^2 = 12x + 20y - 9$.

11. Find the intersections of the straight lines and circles—

(i) $y = 2x - 1$; $x^2 + y^2 - 5x - 3y + 6 = 0$.
(ii) $6x + y = 9$; $x^2 + y^2 - 3x - 7 = 0$.
(iii) $x - 3y = 5$; $x^2 + y^2 - 4x + 22y + 25 = 0$.
(iv) $4y = 3x + 1$; $x^2 + y^2 + 6x + 4y = 12$.
(v) $3x - 4y = 1$; $x^2 + y^2 + 4x + 16y + 18 = 0$.

12. Show that the straight lines touch the circles in each of the following cases, and give the points of contact :—

(i) $x = 2y$; $x^2 + y^2 - 2x - 6y + 5 = 0$.
(ii) $x + y = 1$; $x^2 + y^2 - 8x + 2y + 15 = 0$.
(iii) $x + 2y = 7$; $x^2 + y^2 + 12x + 2y - 8 = 0$.
(iv) $4x - y + 12 = 0$; $x^2 + y^2 - 4x - 6y - 4 = 0$.

13. In cases (i) and (iii) of example 12, verify that the perpendicular to the tangent at the point of contact passes through the centre, and in cases (ii) and (iv) show that the tangent is perpendicular to the diameter through the point of contact.

§§ (VI and VII). 14. Use the formula of corollary 1 of this section to obtain the equation of the chord joining the points (4, 2) and (− 2, 4) on the circle $x^2 + y^2 = 20$. Verify the result by obtaining the equation as in Chap. II.

15. Find the equations of the tangents to the circles at the given points, first verifying that the point is on the circle :—

(i) $x^2 + y^2 = 5$; (2, 1). (v) $x^2 + y^2 = 25$; (4, − 3).
(ii) $x^2 + y^2 = 10$; (1, 3). (vi) $x^2 + y^2 = 34$; (− 3, − 5).
(iii) $x^2 + y^2 = 13$; (3, 2). (vii) $4x^2 + 4y^2 = 25$; $(1\frac{1}{2}, 2)$.
(iv) $x^2 + y^2 = 20$; (− 2, 4). (viii) $9x^2 + 9y^2 = 52$; $(-2, 1\frac{1}{3})$.

16. In each of the following cases find the equation of the chord of contact of the tangents from the point to the circle :—

(i) (1, 2) ; $x^2 + y^2 = 3$. (iii) (8, − 3) ; $x^2 + y^2 = 20$.
(ii) (5, 7) ; $x^2 + y^2 = 6$. (iv) (− 4, − 6) ; $x^2 + y^2 = 25$.

17. Find the co-ordinates of the point of intersection of the tangents drawn to each circle at the extremities of the chord named in the examples below :—

(i) $x^2+y^2=8$; $x+2y=4$. (iii) $x^2+y^2=10$; $2x=4y+5$.
(ii) $x^2+y^2=12$; $2y=x+4$. (iv) $x^2+y^2=20$; $3x+4y+10=0$.

§ (VIII). 18. Find the equation of the tangent to each of the given circles at the point named :—

(i) $x^2 + y^2 - 2x - 10y + 1 = 0$; (6, 5).
(ii) $x^2 + y^2 - 8x + 2y + 15 = 0$; (3, − 2).
(iii) $x^2 + y^2 + 12x + 2y - 8 = 0$; (− 3, 5).
(iv) $x^2 + y^2 - 10x + 8y + 16 = 0$; (2, 0).
(v) $x^2 + y^2 - 3x - 7y + 2 = 0$; (4, 1).
(vi) $x^2 + y^2 - 5x + 9y - 10 = 0$; (− 3, − 2).
(vii) $x^2 + y^2 + 2gx + 2fy$; (0, 0).

§§ (IX and X). 19. Find the length of the tangent drawn from the point to the circle in each of the following cases—

(i) (5, 7) ; $x^2 + y^2 = 6$.
(ii) (2, 3) ; $x^2 + y^2 + 8x + 4y + 8 = 0$.
(iii) (− 3, 4) ; $x^2 + y^2 - 6x - 8y + 5 = 0$.
(iv) (− 5, − 2) ; $x^2 + y^2 + 10x - 4y + 22 = 0$.
(v) (0, 0) ; each of the circles in (ii), (iii) and (iv).

20. Find the equations of the circles on the joins of the following pairs of points as diameters (§ X, Ex. 5) :—

(i) (1, 2) and (5, 6). (iii) (3, 5) and (− 7, − 1).
(ii) (− 2, 4) and (4, − 10). (iv) (0, 0) and (x_1, y_1).

Harder Examples

21. On what circles with O as centre do (2, 3) and (x_1, y_1) lie ?

22. Find the equations of the lines of centres of

(i) $x^2 + y^2 - 8x + 2y - 3 = 0$; $x^2 + y^2 + 4x - 6y + 5 = 0$.
(ii) $x^2 + y^2 - 3x - 7 = 0$; $x^2 + y^2 + 5x + 8 = 0$.

23. Use the results of example 22 to obtain the gradients of the common chords of each of the pairs of circles.

24. Find the distance between the centres of each pair of circles in example 22.

25. A circle passes through the point $(-7, 1)$ and has its centre at the point $(-4, -3)$. Find the length of its radius.

26. Find the equations of the circles passing through the given points and having as centres the points named.

Point	$(-4, -6)$	$(1, 1)$	$(-2, 3)$	$(4, 0)$	$(0, 0)$	$(0, 0)$
Centre	$(0, 0)$	$(2, 3)$	$(1, -4)$	$(0, 3)$	$(-3, 2)$	$(-g, -f)$

27. (i) Find the equation of the line of centres of the two circles
$$x^2 + y^2 - 8x - 4y + 10 = 0$$
and
$$x^2 + y^2 - 10x - 10y + 10 = 0.$$
(ii) Find where the line of centres cuts each circle. What conclusion do you draw ?

28. (i) In what ratio is the join of $(-3, 4)$ and $(7, -1)$ divided by the circle $x^2 + y^2 = 5$? What do you conclude ?
(ii) Where is the line cut ?

29. Find the distance of the point $(2, 1)$ from the centre of the circle $x^2 + y^2 - 6x + 8y + 4 = 0$.
Is the point within or without the circle ?

30. Find which of the points $(1, -2)$, $(3, 4)$ and $(-2, 3)$ are inside the circle $x^2 + y^2 = 16$.

31. Find which of the points $(0, 0)$, $(3, 4)$, $(-1, -11)$ and $(4, -2)$ are outside the circle $x^2 + y^2 - 4x + 10y - 7 = 0$.

32. Find the mid-point of the chord cut off by the circle $x^2 + y^2 = 20$ from the line $x - y = 6$.

33. Show that the lines $3x - 4y = 18$ and $4x - 3y = 17$ intersect on the circle $x^2 + y^2 = 13$.

34. For what values of c will the line $x = 2y + c$ touch the circle $x^2 + y^2 = 5$.

35. Find the equation of the circle with centre O which touches the line $x - 3y + 20 = 0$.

36. Show that $(a \cos \theta, a \sin \theta)$ lies on $x^2 + y^2 = a^2$.

37. Calculate the distance of the line $3x + 4y = 6$ from the centre of the circle $x^2 + y^2 + 6x + 10y - 2 = 0$. Decide from the result whether or not the line cuts the circle.

38. Show that the line $y = x + 2$ cuts each of the circles $x^2 + y^2 + 4x - 12y + 14 = 0$ and $x^2 + y^2 - 14x + 6y - 22 = 0$ in the same two points. Verify the theorem that the common chord is perpendicular to the line of centres and find its length.

39. Show that $2y = x + 11$ is a diameter of the circle $x^2 + y^2 + 6x - 8y - 2 = 0$. Find also the equation of the diameter through O.

40. Show that the lines $5y = 2x + 11$ and $x + 2y = 8$ intersect on the circle $x^2 + y^2 - 4x - y - 2 = 0$.

41. $A \equiv (10, 0)$, $B \equiv (-10, 0)$ and $K \equiv (6, 0)$. The semicircle on AB as diameter meets OY at C. A perpendicular to AB at K meets BC at L. Prove that the circumference bisects KL.

42. The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = 4$ is equal to the distance of the point from the point (5, 6). Prove that the locus of P is a straight line.

43. (i) Find the distance of the line $2x + y = 1$ from the centre of the circle $x^2 + y^2 + 6x - 8y + 3 = 0$.
(ii) Show that the circle cuts off equal chords from the lines $2x + y = 1$ and $11x + 2y + 10 = 0$.

44. Show that the circle on the join of (0, 0) and (a, b) as diameter is the same as that on the join of $(a, 0)$ and $(0, b)$ as diameter. Explain this.

45. Show that the point $(6, -1)$ is equidistant from the points (5, 6) and $(-1, -2)$ and the point $(-5, -3)$ from the points $(-7, -2)$ and $(-3, -4)$.

46. Find the equations of the perpendicular bisectors of the joins of the following pairs of points:—

(i) (1, 1) and (3, 3).
(ii) (2, 7) and (4, 3).
(iii) $(-2, 5)$ and $(6, -7)$.
(iv) $(a, 0)$ and $(0, b)$.

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47. (i) Find where the straight line $x - 2y + 7 = 0$ cuts the circle $x^2 + y^2 - 2x - 10y + 1 = 0$.

(ii) Find the mid-point of the chord cut off.

(iii) Find the equation of the perpendicular bisector of the chord and verify that this line passes through the centre.

48. $A \equiv (3, 0)$, $B \equiv (-3, 0)$ and $P \equiv (x, y)$.

(i) If $PA^2 - PB^2 = 48$, find the locus of P.

(ii) If $PA^2 + PB^2 = 50$, find the locus of P.

49. Find the equations of (i) the tangent and the diameter through the point (2, 5) on the circle $x^2 + y^2 - 6x - 4y + 3 = 0$, and (ii) of the tangent at the other end of the diameter.

50. $A(1, 1)$, $B(2, -2)$ and $C(-6, 2)$ are points on the circle $x^2 + y^2 + 6x + 4y - 12 = 0$. By calculating the angle between (i) AB and BC, (ii) AC and the tangent at A verify a well-known theorem.

51. (i) Show that each of the circles

$$x^2 + y^2 - 12x - 3y - 18 = 0$$

and $x^2 + y^2 + 4x + 9y + 18 = 0$ passes through $(0, -3)$.

(ii) Find the equation of the tangent to each circle at the point named and say what is to be deduced.

(iii) Verify the conclusion by showing that the line of centres passes through $(0, -3)$.

52. $A(-5, 0)$, $B(5, 0)$ and $P(x, y)$ are three points such that $AP : PB = 2 : 3$. Prove that the locus of P is a circle.

53. Show that the tangents at the points (1, 5) and $(-3, 2)$ on the circle $x^2 + y^2 - 4x + y - 27$ make equal angles with the chord of contact.

54. (i) Find the points A, B, C and D where the circle $x^2 + y^2 = 4$ cuts the axes.

(ii) If $P(x_1, y_1)$ be any point on the circumference, prove that $PA^2 + PB^2 + PC^2 + PD^2 = 32$.

55. (i) Write down the conditions that the point (x_1, y_1) lie on each of the circles

$$x^2 + y^2 = 11 \text{ and } x^2 + y^2 - 4x - 5y + 9 = 0.$$

(ii) Deduce that $4x_1 + 5y_1 = 20$ and give the equation of a line on which (x_1, y_1) must lie.

56. Show that if the circles $x^2 + y^2 = 5$ and

$$x^2 + y^2 + 3x + 8y - 20 = 0$$

intersect at the points (x_1, y_1) and (x_2, y_2), then these points must lie on the line $3x + 8y = 15$.

57. (i) Show that the points of intersection (x_1, y_1) and (x_2, y_2) of the circles $x^2 + y^2 - 2x - 10y + 1 = 0$ and $x^2 + y^2 - 3x - 7y + 2 = 0$ lie on the line $x - 3y - 1 = 0$.

(ii) Find now the points of intersection of the circles.

58. If equal tangents can be drawn from the point (x_1, y_1) to each of the circles

$$x^2 + y^2 = 5 \text{ and } x^2 + y^2 - 6x - 4y + 4 = 0,$$

prove that the locus of (x_1, y_1) is a straight line.

59. If the ratio of the tangents from the point (x_1, y_1) to the circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 8x + 3 = 0$ is $1 : 2$, prove that the locus of (x_1, y_1) is a circle.

60. (See § X, Example 4.) Find the circum-circles of the triangles whose vertices are: (i) (0, 0), (4, 0), (1, − 3); (ii) (0,1), (0, 2), (2, 3); (iii) (2, 0), (0, 3), (1, 2); (iv) (2, 5), (6, 1), (0, 3); (v) (1, 2), (2, 3), (4, 1); (vi) (1, 1), (2, − 2), (− 6, 2).

61. If the gradient of the tangent at the point (x_1, y_1) (see § VIII) on the circle $x^2 + y^2 - 4x + 6y - 4 = 0$ is $\frac{1}{4}$, show that (x_1, y_1) lies on the line $4x + y = 5$. Hence find (x_1, y_1).

62. Show that if $x = \dfrac{2t}{1+t^2}$ and $y = \dfrac{1-t^2}{1+t^2}$ then (x, y) is a point on the circle $x^2 + y^2 = 1$. Show also that the tangent at the point is $2tx + (1 - t^2)y = 1 + t^2$.

63. Find the equations of the tangents from the point (4, 7) to the circle $x^2 + y^2 = 1$ (see § X, Example 2).

64. A straight line cuts the axes at $A(3, 0)$ and $B(0, 3)$. From $P(x_1, y_1)$ perpendiculars PK, PL and PM are drawn to AB, BO and OA respectively.

(i) Calculate the length of the perpendicular from P to AB.

(ii) If $PK^2 = PL \,.\, PM$ show that the locus of P is a circle.

65. (i) Show that if the tangent at (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through (x', y'), then (x_1, y_1) lies on $xx' + yy' + g(x + x') + f(y + y') + c = 0$.

(ii) If the tangent at (x_2, y_2) also passes through the point (x', y'), show that (x_2, y_2) also lies on the same line as (x_1, y_1) was shown to lie on.

(iii) What is the equation of the chord of contact of the tangents from (x', y') to the circle ? (See § VII.)

ANSWERS

CHAPTER I

2. (i) $(-2, 0)$; $(0, 1)$. (ii) $(0, 0)$. (iii) $(1, 0)$; $(0, -\frac{1}{2})$. (v) $(2, 0)$; $(0, -\frac{2}{3})$. **3.** $(4{\cdot}5, 2)$. **5.** $(3, \frac{1}{2})$; $(-2, 3)$. **6.** $(2, 3)$, $(3{\cdot}6, 3{\cdot}7)$, $(-1, 1{\cdot}5)$, $(1{\cdot}8, 1{\cdot}9)$, $(-2, 3)$, $(-3{\cdot}3, -1{\cdot}5)$, $(\cdot 85, 1{\cdot}1)$, $(0, 0)$. **7.** $(2{\cdot}8, 3{\cdot}6)$, $(1, 4{\cdot}2)$, $(2{\cdot}5, 2{\cdot}25)$, $(-1{\cdot}5, -2{\cdot}75)$, $(11{\cdot}5, 7)$, $(5{\cdot}5, 10)$, $(-2, 2)$, $(-6, 1{\cdot}5)$. **8.** $\cdot 5$, $25{\cdot}5$, $29{\cdot}5$, 39, $\frac{1}{2}(x_1y_2 - x_2y_1)$. **9.** $(5, 0)$, $(0, 2)$. **10.** $(\pm 3, 0)$, $(0, \pm 2)$. **11.** 3. **12.** $(3, 0)$, $(-3, 6)$. **14.** ± 4. **15.** -4. **16.** $2x_1 + 5y_1 = 7$. **17.** $4x - y = 7$. **18.** $2x^2 - 3xy + y^2 = 10$. **19.** $(9, 4)$. **20.** 3 : 2. **21.** $(3{\cdot}5, 5)$, $(2, 6)$, $(\cdot 5, 4)$, $(2, 5)$. **22.** $(-1, 3)$. **24.** $(1, 6)$. **26.** $(5, -1)$. **27.** 15. **28.** $(3{\cdot}5, 2)$, $(5, 5)$, $(3, 6)$, $(1{\cdot}5, 3)$. **29.** $xy = 5$. **30.** $x^2 = 5y$. **34.** $(2, 0)$, $(6, 0)$, $(0, 3)$, $(0, 4)$; $(4, 3{\cdot}5)$.

CHAPTER II

1. $\frac{1}{3}$, $-\frac{3}{2}$, $-\frac{1}{7}^2$, 1, $\frac{4}{5}$, $\frac{4}{5}$, $\frac{y_1}{x_1}$, $-\frac{b}{a}$. **2.** 18° 26′; 123° 41′; 120° 15′; 45°. **4.** -2, $\frac{2}{3}$, $\frac{5}{6}$, $-\cot\alpha$. **7.** Through O. i, iv, vi, viii; Parallel to an axis. ii, iii, v, vii. **8.** $(3, 2)$, $(2, -1)$, $(-3, -2)$. **9.** (i) 4; $(0, 8)$. (ii) $\frac{2}{3}$; $(0, -6)$. (iii) $-\frac{7}{5}$; $(0, 4)$. (iv) 2; $(0, 0)$. **10.** $y = \frac{3}{2}x - \frac{1}{2}$; $y = \frac{1}{5}x + 1$; $y = \frac{7}{4}x - \frac{1}{4}$; $y = -\frac{3}{5}x - \frac{7}{5}$; $y = 2x + \frac{7}{3}$; $y = -\frac{l}{m}x - \frac{n}{m}$. **11.** $y = \frac{1}{2}x + 3$; $y = 3x - 4$; $y = -\frac{3}{4}x + 6$; $y = -\frac{5}{2}x - 8$. **12.** $\frac{x}{9/4} + \frac{y}{9/2} = 1$; $\frac{x}{-3/5} + \frac{y}{1/2} = 1$; $\frac{x}{5/2} + \frac{y}{-5/3} = 1$; $\frac{x}{2} + \frac{y}{-4} = 1$. **13.** $\frac{x}{-10} + \frac{y}{6} = 1$. **15.** $x - y + 1 = 0$; $x - 2y + 8 = 0$; $2x - 5y = 8$; $x - 6y = 7$; $2x + 7y = -8$; $(b - k)x + hy = bh$. **16.** $3x - 2y = 10$; $4x + 5y + 2 = 0$; $2x - y = 14$; $x + 3y + 30 = 0$. **19.** $(5, 3)$; -9, $-\frac{41}{11}$, $-\frac{23}{6}$, $-\frac{49}{13}$. **20.** $(4, 1)$; $x = 4y$; $x - 5y + 1 = 0$; $x - 2y = 2$; $x = 4$. **21.** $5x = 3y + 12$. **23.** $(0, 0)$, **25.** $y = 1{\cdot}5774x$; $y = \cdot 4227x$; 34° 43′. **27.** $x\cos 52° + y\sin 52° = 6$ or 2; $x\cos 80° - y\sin 80° = -8$ or -2. **28.** $x = 3y$. **29.** $x - y = 2$. **30.** $x + y = 9$. **31.** 1; -1. **32.** $(2{\cdot}5, 1)$; $2x = 5y$. **33.** $4x + 3y = 24$; $(0, +8)$. **34.** $bx + 2ay = 2ab$; $(2a, 0)$. **35.** $x + y = c$. **36.** $(8, 0)$, $(0, -3)$; $3x - 8y = 24$. **37.** $x + 5y = 3$; $(0, \cdot 6)$. **38.** $(-2, 3)$, $(3, -1)$, $(-1, 1)$. **39.** $(3, 2)$; $(2, 3)$; $(2\frac{3}{5}, 2)$. **40.** $(1, 3)$, $(2, 4)$, $(-2\frac{1}{3}, -1\frac{1}{3})$, $(\frac{3}{4}, \frac{3}{2})$. **41.** $x + 2y = 11$; $4x - 5y = 23$. **42.** $7x + 5y = 12$; $9x = 7y$. **43.** (i) $y = mx$; $bx + ay = 0$; $x\cos\alpha + y\sin\alpha = 0$. (ii) $x + my = 0$; $ax - by = 0$; $x\cos\alpha - y\sin\alpha = 0$. **44.** $x + y = c$; $x = y$. **45.** $2x - 3y = 19$. **46.** $(0, 0)$, $(3, 3)$, $(1, 5)$, $(-2, 2)$. **49.** $(4, 2)$; $\left(-\frac{bm}{m^2+1}, \frac{b}{m^2+1}\right)$. **50.** $hx + ky = h^2 + k^2$; $\left(\frac{h^2+k^2}{h}, \frac{h^2+k^2}{k}\right)$. **51.** (i) $3x + 2y = 32$; $2x - y = 5$; $x + 3y = 27$. (ii) $9y = 2x + 5$; $x + 1 = 0$; $2x + 9y = 1$. **57.** $(2{\cdot}8, 0{\cdot}2)$. **63.** $2y = 5x + 9$. **64.** $(1, 2)$. **65.** $(3, 6)$, $(27, 18)$.

CHAPTER III

1. 10° 18′; 40° 14′; 2° 56′. **2.** 47° 44′. **3.** 22° 23′; 2° 45′; 79° 57′; 85°; 0°; 90°. **4.** Opposite **5.** Origin side $(1, 0)$, $(3, -5)$. **6.** R.H.S. $(-1, 4)$, $(2, 3)$. **7.** 3·2; 2; 5; 1; 1, $\frac{2}{13}$, $\frac{1}{5}$, $\frac{3}{13}$;

5; $\frac{b}{1+m^2}$; $\frac{ab}{a^2+b^2}$; $p - \sin\alpha - \cos\alpha$; $\sqrt{m^2+1}$; $\sqrt{m^2+1}$; $\sqrt{l^2+m^2}$. **8.** $x-y=0, x+y=0$; $16x=2y+49, x+8y+1=0$; $6x-7y+2=0, 7x+6y=26$; $21x-y+21=0, x+21y+1=0$; $\sqrt{17}(2x-3y)=\pm\sqrt{13}(x+4y)$; $x=0, y=0$. **9.** Point on line. **10.** 4·8. **11.** 12·8. **12.** $\frac{ab}{a+b}$. **14.** $4x+3y+10=0$. **15.** $3x+4y=31$, $3x+4y+19=0$. **18.** 2; 2; $p_1 \pm p_2$; $\frac{b \pm c}{\sqrt{1+m^2}}$. **20.** 2·4. **22.** 2. **23.** 5. **24.** ± 60. **25.** $3x+4y+10=0$; $3x+4y=30$; $4x=3y-15$; $4x=3y+25$. **26.** $\sqrt{{y_1}^2+64}$ by $\frac{48}{\sqrt{{y_1}^2+64}}$.

CHAPTER IV

N.B.—Equations like $x^2+y^2+2gx+2fy+c=0$ will be printed as follows: $\blacksquare + 2gx + 2fy + c = 0$.

1. 5, 13, 17, $\sqrt{13}$, $\sqrt{17}$, $\sqrt{29}$. **2.** $x^2+y^2=1$, 9, 49. **3.** 2, 3, 6, $\frac{5}{2}$, $\frac{8}{3}$. **4.** (2, 4), (4, 2); (1, 3), (− 1·8, − 2·6); (2, 2), (− ·4, 2·8); (3, 2), (− 2·76, − 2·32). **5.** (2, − 1); (2, 3). **6.** 2, $\frac{1}{2}$; 1, $-\frac{1}{3}$; 2, − 2; $\frac{1}{2}$, $-\frac{11}{10}$; $\pm 2\sqrt{\frac{2}{21}}$. **7.** 5, 13, 20, 2·9, 5·1, 20, 5c. **8.** $\blacksquare - 8x - 4y + 11 = 0$; $\blacksquare + 12x - 10y + 57 = 0$; $\blacksquare + 6x + 8y = 0$; $\blacksquare - 6y - 40 = 0$; $\blacksquare - 2ax - 2ay + a^2 = 0$. **9.** Nos. i, iii, iv, vi. **10.** (4, 3) (2); (2, − 5) (6); (5, 0) (4); (− 12, 5) (13); ($3\frac{1}{2}$, 3) ($4\frac{1}{2}$); (− $\frac{1}{2}$, 1) ($\frac{1}{2}$); (1, − $1\frac{1}{2}$) ($2\frac{1}{2}$); ($2\frac{1}{2}$, − 3) (4); (− $\frac{1}{2}$, $\frac{1}{4}$) ($2\frac{1}{2}$); ($1\frac{1}{2}$, + $2\frac{1}{2}$) ($2\frac{1}{2}$). **11.** (1, 1), (2, 3); (1, 3), (2, − 3); (2, − 1), (− 4, − 3); (1, 1), (− 7, − 5); (− 1, − 1), (− 9, − 7). **12.** (2, 1); (3, − 2); (− 3, 5); (− 2, 4). **15.** $2x+y=5$; $x+3y=10$; $3x+2y=13$; $2y=x+10$; $4x-3y=25$; $3x+5y+34=0$; $6x+8y=25$; $6y=9x+26$. **16.** $x+2y=3$; $5x+7y=6$; $8x-3y=20$; $4x+6y+25=0$. **17.** (2, 4); (− 3, 6); (4, − 8); (− 6, − 8). **18.** $x=6$; $x+y=1$; $x+2y=7$; $3x-4y=6$; $x-y=3$; $5y=11x+23$; $gx+fy=0$. **19.** $\sqrt{68}$; 7; 4; 3; $\sqrt{8}$, $\sqrt{5}$, $\sqrt{22}$. **20.** $\blacksquare - 6x - 8y + 17 = 0$; $\blacksquare - 2x + 6y - 48 = 0$; $\blacksquare + 4x - 4y - 26 = 0$; $\blacksquare = xx_1+yy_1$. **21.** $x^2+y^2=13$, ${x_1}^2+{y_1}^2$. **22.** $2x+3y=5$; $y=0$. **23.** $\frac{3}{2}$; ∞. **24.** $2\sqrt{13}$; 4. **25.** 5. **26.** $x^2+y^2=52$; $\blacksquare - 4x - 6y + 8 = 0$; $\blacksquare - 2x + 8y - 41 = 0$; $\blacksquare - 6y - 16 = 0$; $\blacksquare + 6x - 4y = 0$; $\blacksquare + 2gx + 2fy = 0$. **27.** $y=3x-10$; (3, − 1), (5, 5); (3, − 1), (7, 11). **28.** 2 : 3; Tangent; (1, 2). **29.** $\sqrt{26}$; outside. **30.** (1, − 2). **31.** (3, 4). (− 1, − 11). **32.** (3, − 3). **34.** ± 5. **35.** $x^2+y^2=40$. **37.** 7; Not. **38.** $4\sqrt{2}$. **39.** $4x+3y=0$. **43.** $\frac{3}{\sqrt{5}}$. **46.** $x+y=4$; $2y=x+7$; $2x-3y=7$; $2ax-2by=a^2-b^2$. **47.** (−3, 2), (5·8, 6·4); (1·4), (4·2); $2x+y=7$. **48.** $x=-4$; $x^2+y^2=16$. **49.** $3y=x+13$; $3x+y=11$; $x=3y+7$. **51.** $4x+3y+9=0$. **54.** ($\pm$ 2, 0), (0 $\pm$ 2). **55.** $4x+5y=20$. **57.** (1, 0), (4, 1). **60.** $\blacksquare - 4x + 2y = 0$; $\blacksquare - 3x - 3y + 2 = 0$; $\blacksquare + 7x + 3y - 18 = 0$; $\blacksquare - 6x - 4y + 3 = 0$; $\blacksquare - 5x - 3y + 6 = 0$. **61.** (1, 1), (3, −7). **63.** $12x=5y+13$; $4x=3y-5$.

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